Math 2321 (Multivariable Calculus)

Lecture #35 of 37 $\sim$ April 15, 2021

Applications of Multivariable Calculus, Part 1

- Planetary Motion, Orbits, and Gravity
- Some Miscellaneous Applications

Logistical Things

- I've applied all of the remaining WeBWorK extensions to set 12 for everyone.
- The final exam is 2 hours, with some additional turnaround time added like with the midterms.
- The final exam will be held from 10:30am-12:30pm on Thursday, April 29th, unless you have made alternate arrangements due to an exam conflict.
- The best way to study for our final is to practice with the old finals. Next Wednesday I will do some review problems. I will also run a 2-hour review session during the final exam week, date and time TBD by Piazza vote.
- If your score on the final is better than any of your midterm scores (as a percentage), then your lowest midterm exam score is dropped.

What Have We Learned?

Here's a summary of the material we covered this semester:

1. Vectors and 3D Geometry: graphs and level sets, vectors, dot and cross products, lines and planes, vector-valued functions, curves and motion in 3-space.
2. Partial Derivatives: limits and partial derivatives, directional derivatives, gradients, tangent lines and planes, the chain rule, linearization, minima and maxima, classifying critical points, optimization on a region, Lagrange multipliers.
3. Multiple Integration: double and triple integrals in rectangular, polar, cylindrical, and spherical coordinates, area, volume, average value, mass, center of mass.
4. Vector Calculus: line and surface integrals, vector fields, work, circulation, flux, conservative fields and potential functions, the fundamental theorem of line integrals, divergence and curl, Green's theorem, Stokes's theorem, the divergence theorem.

Motion, Orbits, and Gravity, I

We will start out by discussing planetary motion using some of our properties of vectors and vector fields.

- The main tool we will use for our analysis is Newton's law of gravitation.
- Newton's law of gravitation says that the gravitational attraction imparted by an object on a particle is directly proportional to each of their mass and inversely proportional to the square of the distance between them.
- If we have a list of objects along with their masses and locations, we can use Newton's law of gravitation to write down the vector field modeling the force due to gravity at a given point (x, y, z).
- The simplest situation is with an object and a particle.

Motion, Orbits, and Gravity, II

Explicitly, suppose the particle has mass m at $\mathbf{r} = \langle x, y, z \rangle$, and the object has mass M and is located at the origin.

- The direction of the gravitational field is from $\mathbf{r}$ to the origin $(0, 0, 0)$. The unit vector in this direction is $-\dfrac{\mathbf{r}}{||\mathbf{r}||}$.
- The magnitude of the field $\mathbf{F}$ is equal to a constant times m times M times $\dfrac{1}{||\mathbf{r}||^2}$. The constant here is called G, the universal gravitational constant, and its value has been measured to be $6.674\,\mathrm{m}^3/(\,\mathrm{kg}\cdot\mathrm{s}^2)$.
- Thus, we see $\mathbf{F} = \dfrac{GmM}{||\mathbf{r}||^2} \cdot \left(-\dfrac{\mathbf{r}}{||\mathbf{r}||} \right) = -\dfrac{GmM}{||\mathbf{r}||^3}\mathbf{r}$.
- As a function of x, y, z, this is $\mathbf{F}(x, y, z) = -\dfrac{GmM}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle$.

Motion, Orbits, and Gravity, III

Now suppose that the path $\mathbf{r}$ describes the motion of a planet (mass m) through space, and the only force acting on the planet is the gravity of the sun (mass M) at the origin. We claim that the planet's orbit lies in a plane passing through the sun.

- First, note that the gravitational force is $\mathbf{F}(\mathbf{r}) = -\dfrac{GmM}{||\mathbf{r}||^3}\mathbf{r}$.
- By Newton's second law ($\mathbf{F} = m\mathbf{a}$), the particle's acceleration satisfies $\mathbf{a}(r) = -\dfrac{GM}{||\mathbf{r}||^3}\mathbf{r}$, which is a scalar multiple of $-\mathbf{r}$.

Motion, Orbits, and Gravity, IV

We can now show that the planet's orbit lies in a plane by showing that $\mathbf{n} = \mathbf{r} \times \mathbf{v}$ is constant.

- If this is true, then the position and velocity both lie in the plane whose normal vector is $\mathbf{n}$, and so the particle's motion will stay in the plane.
- To show $\mathbf{n} = \mathbf{r} \times \mathbf{v}$ is constant, we calculate its derivative: $\frac{d}{dt}[\mathbf{r} \times \mathbf{v}] = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}$.
- The first cross product is zero, since the cross product of any vector with itself is zero, and the second cross product is also zero, since $\mathbf{a}$ is a scalar multiple of $\mathbf{r}$ as noted on the last slide.
- Thus, $\frac{d}{dt}[\mathbf{r} \times \mathbf{v}]$ is zero, and so $\mathbf{n} = \mathbf{r} \times \mathbf{v}$ is a constant vector.
- Therefore, the planet's orbit lies in a plane, as claimed.

Motion, Orbits, and Gravity, V

By extending this sort of calculation, one may derive Kepler's famous laws of planetary motion:

1. The orbit of a planet is a conic section with the sun at one focus. Specifically, the conic's eccentricity is $e = \frac{r_0 v_0^2}{GM} - 1$.
2. The radius vector $\mathbf{r}$ from the sun to the planet sweeps out equal areas in equal times.
3. The square of the orbital period T is proportional to the cube of the length of the semimajor axis a. Specifically, $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$.

To show Kepler's laws, the best approach is to work in polar coordinates. This approach is feasible because, as we have just shown, the orbit of a planet lies in a plane passing through the sun. To show you what this looks like, I will work out the second law. For notational convenience I will use dots to denote time-derivatives.

Motion, Orbits, and Gravity, VI

Place the sun at the origin and define $r = ||\mathbf{r}||$ to be the radial parameter, with angle parameter θ.

- Then define the unit vector $\mathbf{u}_r = \langle \cos\theta, \sin\theta, 0 \rangle$, which is the unit vector in the direction of increasing r, and the orthogonal unit vector $\mathbf{u}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$, which is the unit vector in the direction of increasing θ.
- We see $\dfrac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta$ and $\dfrac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r$, so the chain rule gives $\dot{\mathbf{u}}_r = \dfrac{d\mathbf{u}_r}{d\theta}\dfrac{d\theta}{dt} = \dot{\theta}\mathbf{u}_\theta$ and $\dot{\mathbf{u}}_\theta = \dfrac{d\mathbf{u}_\theta}{d\theta}\dfrac{d\theta}{dt} = -\dot{\theta}\mathbf{u}_r$.
- Then because $\mathbf{r} = r\mathbf{u}_r$, by the product rule we get $\mathbf{v} = \dot{\mathbf{r}} = \dfrac{dr}{dt}\mathbf{u}_r + r\dfrac{d\mathbf{u}_r}{dt} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$.

Motion, Orbits, and Gravity, VII

We can then derive Kepler's second law by calculating the vector $\mathbf{n} = \mathbf{r} \times \mathbf{v}$ from earlier.

- Explicitly, we have $\mathbf{n} = (r\mathbf{u}_r) \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = r\dot{r}(\mathbf{u}_r \times \mathbf{u}_r) + r^2\dot{\theta}(\mathbf{u}_r \times \mathbf{u}_\theta) = \left\langle 0, 0, r^2\dot{\theta} \right\rangle$.
- Now we can compute the area swept out by the radius vector between time $t = t_1$ and time $t = t_2$. By integrating in polar and then doing a substitution in the resulting line integral, this is $\int_{\theta_1(t)}^{\theta_2(t)} \int_0^{r(t)} 1 \cdot r\,dr\,d\theta = \int_{\theta_1(t)}^{\theta_2(t)} \frac{1}{2} r^2\,d\theta = \int_{t_1}^{t_2} \frac{1}{2} r(t)^2 \dot{\theta}(t)\,dt$.
- However, the integrand in the last integral is exactly the z-component of the constant vector $\mathbf{n}$, so the integral is simply $(t_2 - t_1)$ times a constant. This means the area depends only on the amount of time $t_2 - t_1$, which (when phrased more elegantly) is Kepler's second law.

Motion, Orbits, and Gravity, VIII

The other laws are a bit more difficult and require careful manipulation of the differential equation $\mathbf{r}''(t) = -\frac{GmM}{||\mathbf{r}||^3}\mathbf{r}$.

- I won't go through all the details of these.
- But the first law boils down to computing the polar equation for r in terms of θ, and verifying it has the form $r = \dfrac{(1+e)r_0}{1+e\cos\theta}$ where r_0 is the radius at perihelion (i.e., the minimal radius), which we take to occur at $t = 0$ and $\theta = 0$.
- The third law boils down to comparing two formulas for the area of an ellipse (one of them is the integral formula from the second law, and the other is π times the semimajor axis times the seminor axis, which we showed as an application of Green's theorem).

The Gravity of The Situation, I

We can do a little bit more with Newton's law of gravitation

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{||\mathbf{r}||^3}\mathbf{r} = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle.$$

- Specifically, we often want to calculate the work done by a gravitational field on an object.
- Calculating the work integral as a line integral directly is messy because of the square root factor in the denominator.
- But we might hope that **F** is conservative. In fact, it is! (It is not so fun to compute $\operatorname{curl} \mathbf{F}$, but it is zero.)

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- Specifically, we often want to calculate the work done by a gravitational field on an object.
- Calculating the work integral as a line integral directly is messy because of the square root factor in the denominator.
- But we might hope that $\mathbf{F}$ is conservative. In fact, it is! (It is not so fun to compute $\operatorname{curl}\mathbf{F}$, but it is zero.)
- If we search for a potential function, we can eventually see that $U = GmM(x^2+y^2+z^2)^{-1/2}$ has the property that $\mathbf{F} = \langle U_x, U_y, U_z\rangle$: the chain rule terms $2x$, $2y$, $2z$ exactly give the needed factors of x, y, z in the three components.

The Gravity of The Situation, II

So, we see that the gravitational field $\mathbf{F}(\mathbf{r}) = -\frac{GmM}{||\mathbf{r}||^3}\mathbf{r} = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$ is conservative and has a potential function $U = GmM(x^2+y^2+z^2)^{-1/2} = GmM/||\mathbf{r}||$.

- We can therefore easily compute the work done by the vector field on a particle that travels from point **a** to point **b**.
- Specifically, by the fundamental theorem of line integrals, the work done by **F** is equal to $U(\mathbf{b}) - U(\mathbf{a}) = \frac{GmM}{||\mathbf{b}||} - \frac{GmM}{||\mathbf{a}||}$.
- Notice that the work only depends on the distances of the points from the origin.

The Gravity of The Situation, III

We can even apply this to estimate the gravitational potential energy on the surface of the Earth.

- The change in the potential energy function is $U(\mathbf{b}) - U(\mathbf{a}) = \frac{GmM}{||\mathbf{b}||} - \frac{GmM}{||\mathbf{a}||}$.
- If the start and end points are both approximately a distance R from the origin, then we can estimate the change in the potential energy using a linearization (or, if you prefer, a directional derivative).

The Gravity of The Situation, IV

Specifically, the linearization of $U(x) = \frac{GmM}{x}$ at $x = R$ is $L(x) = \frac{GmM}{R} - \frac{GmM}{R^2}(x - R)$.

- Therefore, the approximate value of $U(R + \Delta h) - U(R)$ is $-\dfrac{GmM}{R^2}\Delta h$, which equals $mg\Delta h$ where m is the mass of the particle, h is the change in height, and $g = \dfrac{GM}{R^2}$ is a constant.
- If we evaluate this constant g using the known values $G = 6.674\,\mathrm{m}^3/(\mathrm{kg}\cdot \mathrm{s}^2)$, the mass of the Earth $M = 5.972 \cdot 10^{24}\,\mathrm{kg}$, and the radius of the Earth $R = 6.371 \cdot 10^6\,\mathrm{m}$, we obtain (drumroll)

The Gravity of The Situation, IV

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- If we evaluate this constant g using the known values $G = 6.674\,\text{m}^3/(\text{kg}\cdot\text{s}^2)$, the mass of the Earth $M = 5.972 \cdot 10^{24}\,\text{kg}$, and the radius of the Earth $R = 6.371 \cdot 10^6\,\text{m}$, we obtain (drumroll) the local gravitational constant $g = 9.817\,\text{m}/\text{s}^2$.
- This should not be a surprise, because by Newton's second law, the magnitude of the acceleration due to the gravitational field will be $||\mathbf{F}(\mathbf{r})||/m = -GM/||\mathbf{r}||^2$.

Some Miscellaneous Applications, I

[Conservative Fields] Recall that we defined conservative vector fields as those where the work is independent of the path, and we showed that this is equivalent to saying that the circulation around any closed curve C is zero.

- If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, then by Stokes's theorem, for any surface S whose boundary is C, we have $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = 0$.
- Thus, the integral of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ is zero on any surface S.
- But if this quantity is zero on every surface, then in fact the curl $\nabla \times \mathbf{F}$ must be zero: otherwise, if it took a nonzero value $\mathbf{v}$ at a point P, we could pick a small patch of a plane passing through P with normal vector $\mathbf{v}$, and then $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = ||\mathbf{v}||^2$ would be positive, meaning that its integral could not be zero.
- Conversely, again by Stokes's theorem, if $\nabla \times \mathbf{F} = 0$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = 0$, and so $\mathbf{F}$ is conservative.

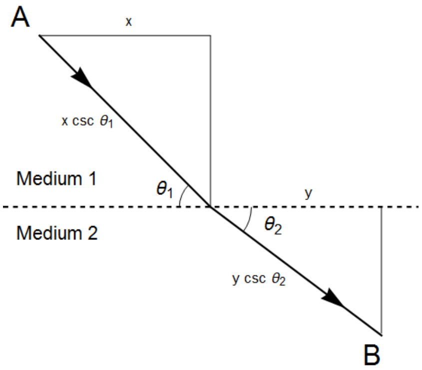
Some Miscellaneous Applications, II

[Economics] The Wilson lot size formula computes the optimal amount of a product to keep on hand based on its various costs.

- If the purchase price for an item is P with annual demand D, the cost of ordering one shipment is K, and the holding cost for an item is h.
- Then the total expected cost is $T = PD + KD/Q + hQ/2$ where Q is the number of items.
- We can minimize T with respect to Q by computing $\partial T/\partial Q$ and setting it equal to zero: this yields $\partial T/\partial Q = -KD/Q^2 + h/2$, so $Q = \sqrt{2KD/h}$.
- If we want to understand how this function will change with a change in the parameters (e.g., if the annual demand D goes up, or the ordering cost K goes down), we can then simply calculate the appropriate partial derivatives, or more generally a directional derivative, of $Q = \sqrt{2KD/h}$.

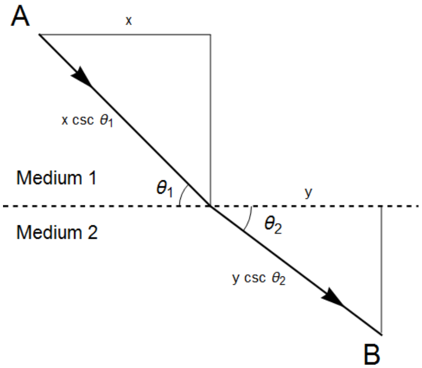
Some Miscellaneous Applications, III

[Optics] Snell's law characterizes how waves will refract when passing from one medium to another: if θ_1 is the angle of incidence and θ_2 is the angle of refraction, then $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_2}{v_1}$, where v_i and v_r are the wave speeds in the two different media.



- We can deduce Snell's law using Fermat's principle: the wave follows the path that takes the least total amount of time.
- For points A and B, the total time taken is $x\csc\theta_1/v_1 + y\csc\theta_2/v_2$, with constraint $x + y = a$.

Some Miscellaneous Applications, IV



- The total time taken is $x \csc \theta_1 / v_1 + y \csc \theta_2 / v_2$, with constraint $x + y = a$.
- Now we can use Lagrange multipliers with $f(x, y) = x \csc \theta_1 / v_1 + y \csc \theta_2 / v_2$ and $g(x, y) = x + y = a$.

- The condition $\nabla f = \lambda \nabla g$ is $\langle \csc \theta_1 / v_1, \csc \theta_2 / v_2 \rangle = \lambda \langle 1, 1 \rangle$.
- Thus, $\dfrac{\csc \theta_1 / v_1}{\csc \theta_2 / v_2} = 1$ which is the same as $\dfrac{\sin \theta_1}{\sin \theta_2} = \dfrac{v_2}{v_1}$.

Some Miscellaneous Applications, V

[Fluid Dynamics] Archimedes' principle says that if a body is immersed in a fluid, the net effect of the fluid pressure on the surface of the body (the buoyant force) is vertical and equals the weight of fluid displaced by the body.

- We can deduce Archimedes' principle using the divergence theorem.
- Fluid pressure on the solid D acts on the solid's surface S perpendicularly to the surface inward.
- Assume that the fluid fills the region with $z \le 0$, with gravity acting in the z-direction.
- By Pascal's law, the pressure of the fluid equals the total mass of fluid directly above a given point.
- Thus, the pressure of the fluid at (x, y, z) is $\langle 0, 0, -\rho z \rangle$, where ρ is the density of the fluid. (If you like, we could even compute the mass with a line integral.)

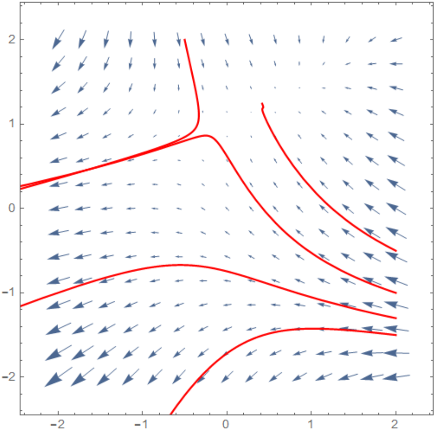
Some Miscellaneous Applications, VI

The pressure of the fluid at (x, y, z) is $\langle 0, 0, \rho z \rangle$, where ρ is the density of the fluid.

- Since the pressure is zero in the x and y-directions, the horizontal pressure on any body is always zero.
- The force component in the vertical direction is given by the surface integral $-\iint_S \langle 0, 0, -\rho z \rangle \cdot \mathbf{n}\, d\sigma$, where $\mathbf{P}$ is the vector field giving the fluid pressure at a given point. (The minus sign is because pressure pushes inward, not outward).
- By the divergence theorem, this surface integral equals $-\iiint_D \mathrm{div}(0, 0, -\rho z)\, dV = \iiint_D \rho\, dV$, which is exactly the volume of the solid times the density ρ of the fluid.
- Thus, the total force is directed upward, and its magnitude equals the volume of the displaced fluid: this is Archimedes' principle.

Some Miscellaneous Applications, VII

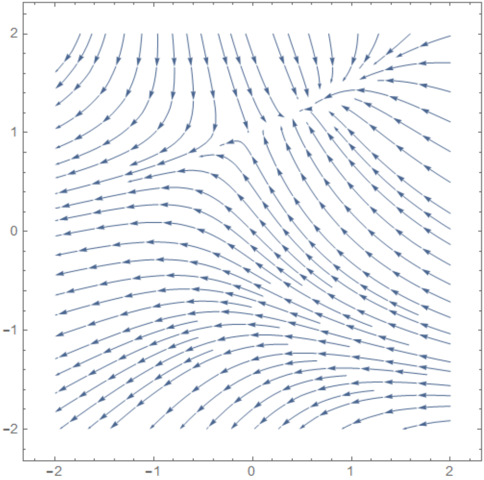
[Streamlines] If $\mathbf{F}(x, y)$ represents the flow of a fluid in the plane, we can trace the path of a particle moving through the fluid by following the vector field $\mathbf{F}(x, y)$ at every point.



- These paths are called <u>streamlines</u>.
- They represent solutions to the system of differential equations $\langle x'(t), y'(t)\rangle = \mathbf{F}(x(t), y(t))$.
- Such systems are common in applications like ecology (e.g., predator-prey systems) and engineering (mixing of substances).

Some Miscellaneous Applications, VIII

Here is a slightly different way of plotting streamlines:



- For some vector fields, there will be streamlines that form closed curves (i.e., loops).
- Other vector fields will not have any closed streamlines.
- Using Green's theorem, we can give a criterion for when a vector field will not have closed streamlines.

Some Miscellaneous Applications, IX

Bendixon's criterion says that a vector field $\mathbf{F} = \langle P, Q \rangle$ will not have any closed streamlines if $P_x + Q_y$ is never zero.

- To see this, suppose we did have a closed streamline C, and consider the flux of the vector field $\mathbf{F}$ across C.
- Since the vector field $\mathbf{F}$ always points in the tangential direction along C, the component in the normal direction is zero. Therefore, the flux integral $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds$ is zero.
- But by the normal form of Green's theorem, the flux is also equal to $\iint_R \operatorname{div}(\mathbf{F})\, dA = \iint_R (P_x + Q_y)\, dA$.
- But if $P_x + Q_y$ is never zero, then (since it is continuous) it is either always positive or always negative – and then the double integral would have the same property.
- This is impossible, so we can't have a closed streamline C.

Summary

We did some more examples of problems involving Stokes's theorem and the divergence theorem.

We discussed some applications of multivariable calculus to analyzing planetary motion, orbits, and gravity.

We discussed some other miscellaneous applications of multivariable calculus to economics, optics, and fluid dynamics.

Next lecture: More applications of vector calculus