Math 2321 (Multivariable Calculus)

Lecture #34 of 37 $\sim$ April 14, 2021

Stokes's Theorem and the Divergence Theorem

- Stokes's Theorem
- The Divergence Theorem

This material represents §4.6 from the course notes. This is the last new material for the semester!

Tangential Form of Green's Theorem

We first discuss the generalization of the tangential form of Green's theorem. Here is a reminder of that version of Green's theorem:

Theorem (Green's Theorem, Tangential Form)

If C is a simple closed rectifiable curve in the plane oriented counterclockwise around the boundary of the region R, then the circulation around C is given by

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$

where $\mathbf{T}$ is the unit tangent to the curve and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Stokes's Theorem, I

And now here is our 3-dimensional generalization of the tangential form of Green's theorem:

Theorem (Stokes's Theorem)

If C is a simple closed rectifiable curve in 3-space that is oriented counterclockwise around the boundary of the surface S, then the circulation around C is given by

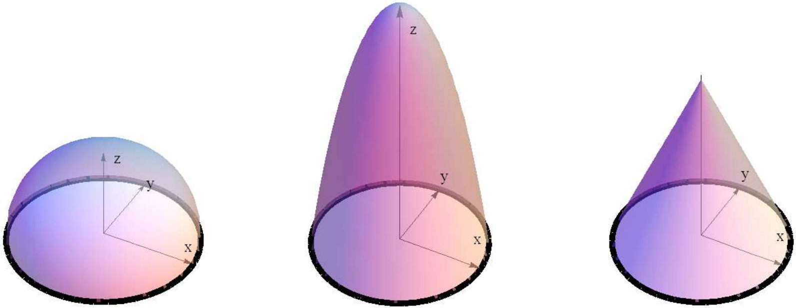
$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{T}$ is the unit tangent to the curve and $\mathbf{n}$ is the unit normal to the surface.

The tangential form of Green's theorem is the special case of Stokes's theorem where the curve and "surface" are in the xy-plane (then the normal vector is simply the vector $\mathbf{k}$).

Stokes's Theorem, II

But, unlike in Green's theorem where a given curve encloses a unique possible region, there are many possible surfaces that a given curve C can bound. Here are some surfaces whose boundaries are the the circle $x^2 + y^2 = 1$, $z = 0$ in the xy-plane:



Stokes's Theorem, III

A few remarks about Stokes's theorem:

- The curve C must run counterclockwise around S: in other words, when walking along C, the surface should be on its left-hand side.
- The unit normal vector to S is oriented via the right-hand rule: using your right hand, curl your fingers along C: your thumb points in the proper direction for $\mathbf{n}$.
- If you want the curve to run clockwise around a surface, that is equivalent to traversing the curve in the opposite direction, and so the integral will be scaled by -1.
- The hypotheses about the curve ("simple, closed, rectifiable, oriented counterclockwise") are the same as in Green's Theorem, and they ensure the curve is nice enough for the theorem to hold.

Stokes's Theorem, IV

A few more remarks about Stokes's theorem:

- Intuitively, if we think of a vector field as modeling the flow of a fluid, the quantity $(\text{curl}\,\mathbf{F}) \cdot \mathbf{n}$ at (x, y, z) measures how much the fluid is circulating around the point (x, y, z) along the surface.
- Stokes's Theorem then says: we can measure how much the fluid circulates around the whole surface by measuring how much it circles around its boundary.
- The proof of Stokes's Theorem (which we omit) can essentially be reduced to the proof of Green's Theorem: if we parametrize the surface and break it into patches, then Stokes's Theorem follows by applying the tangential form of Green's Theorem on each patch and then summing over the patches.

Stokes's Theorem, V

Stokes's theorem gives an equality

$$\text{Circulation around } C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

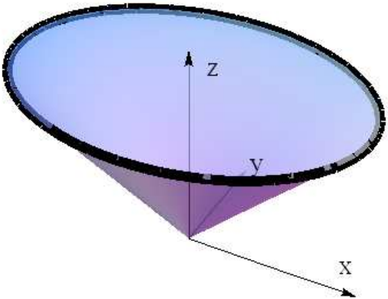
- Typically, we use Stokes's Theorem when the line integral over the boundary is difficult, but there is a nicer surface available.
- However, sometimes we can use the theorem in the other direction, if we happen to be computing a surface integral that involves the curl of a vector field.

Stokes's Theorem, VI

Example: Find the circulation of the vector field $\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ around the ellipse given by the intersection of the upper half of the ellipsoid $x^2 + 2y^2 + 2z^2 = 12$ with the cone $x^2 + 2y^2 = z^2$.

Stokes's Theorem, VI

Example: Find the circulation of the vector field $\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ around the ellipse given by the intersection of the upper half of the ellipsoid $x^2 + 2y^2 + 2z^2 = 12$ with the cone $x^2 + 2y^2 = z^2$.



- We could find a parametrization for the ellipse (it has $x^2 + 2y^2 = 4$ and $z = 2$) and then set up the circulation integral.
- However, this is quite messy, since it will involve large powers of sines and cosines.

Stokes's Theorem, V

Example: Find the circulation of the vector field $\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ around the ellipse given by the intersection of the upper half of the ellipsoid $x^2 + 2y^2 + 2z^2 = 12$ with the cone $x^2 + 2y^2 = z^2$.

- Another way is to try to use Stokes's Theorem.

Stokes's Theorem, V

Example: Find the circulation of the vector field $\mathbf{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ around the ellipse given by the intersection of the upper half of the ellipsoid $x^2 + 2y^2 + 2z^2 = 12$ with the cone $x^2 + 2y^2 = z^2$.

- Another way is to try to use Stokes's Theorem.
- Since the curve runs counterclockwise around the ellipsoid, we will use that as the surface.
- We know that
  Circulation around $C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, d\sigma$.
- We have $\text{curl}\, \mathbf{F} =$
  $\langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3\rangle = \langle 0, 0, 0\rangle$.
- So the curl of $\mathbf{F}$ is zero. Hence $(\text{curl}\, \mathbf{F}) \cdot \mathbf{n}$ will also be zero, so we see that the circulation is 0, without even having to set up the surface integral.

Stokes's Theorem, VI

In this last example, we applied Stokes's theorem to calculate the circulation of a vector field whose curl was zero.

- However, we could have also solved this problem by noting that the vector field was conservative, and thus we could have computed a potential function.
- Then the circulation integral would automatically be zero, because the start and end points of the curve are the same.
- In fact, Stokes's theorem is actually the result that underlies this entire method to begin with!
- By this simple application of Stokes's theorem, we can actually *deduce* this fact (which, if you recall, I didn't fully prove when we discussed conservative fields) that a vector field with zero curl is always conservative.

Stokes's Theorem, VII

Example: Find the circulation of $\mathbf{F}(x, y, z) = \langle 2xyz, x^2z, 2x^2y \rangle$ around the counterclockwise boundary of the portion of the surface $z = xy$ with $0 \le x \le 1$ and $0 \le y \le x$.

- We could parametrize the boundary of this surface, but it will have three components and is rather complicated.
- Instead, we can use Stokes's theorem to convert the circulation integral into a surface integral on the given surface.
- We have

$$\text{Circulation around } C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dA$$

so we just need to set up the surface integral.

Stokes's Theorem, VIII

Example: Find the circulation of $\mathbf{F}(x, y, z) = \langle 2xyz, x^2 z, 2x^2 y \rangle$ around the counterclockwise boundary of the portion of the surface $z = xy$ with $0 \leq x \leq 1$ and $0 \leq y \leq x$.

- We can parametrize the surface as $x = s$, $y = t$, $z = st$ for $0 \leq t \leq s$, $0 \leq s \leq 1$.

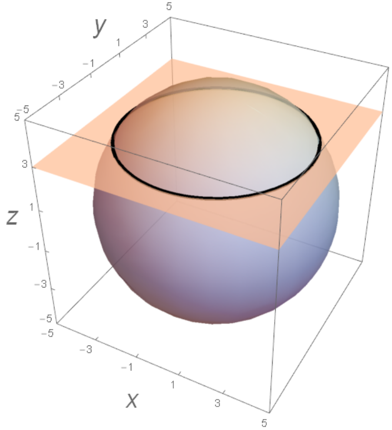
Stokes's Theorem, VIII

Example: Find the circulation of $\mathbf{F}(x, y, z) = \left\langle 2xyz, x^2 z, 2x^2 y \right\rangle$ around the counterclockwise boundary of the portion of the surface $z = xy$ with $0 \le x \le 1$ and $0 \le y \le x$.

- We can parametrize the surface as $x = s$, $y = t$, $z = st$ for $0 \le t \le s$, $0 \le s \le 1$. Then $\mathbf{r}(s,t) = \langle s, t, st \rangle$ and so $(d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 1, 0, t \rangle \times \langle 0, 1, s \rangle = \langle -t, -s, 1 \rangle$.
- The orientation here is correct since the z-component is positive. Also, $\operatorname{curl} \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle = \langle x^2, -2xy, 0 \rangle = \langle s^2, -2st, 0 \rangle$.
- Thus, by Stokes's theorem, the circulation is
$$\iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^1 \int_0^s \langle -t, -s, 1 \rangle \cdot \langle s^2, -2st, 0 \rangle dt\, ds$$
$$= \int_0^1 \int_0^s s^2 t\, dt\, ds = \int_0^1 \frac{1}{2} s^4\, ds = \frac{1}{10}.$$

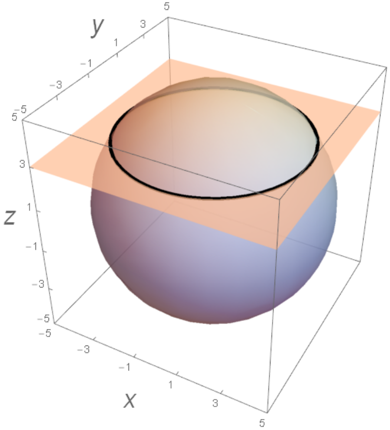
Stokes's Theorem, IX

Example: Find the flux of the curl $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the portion of $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward unit normal.



Stokes's Theorem, IX

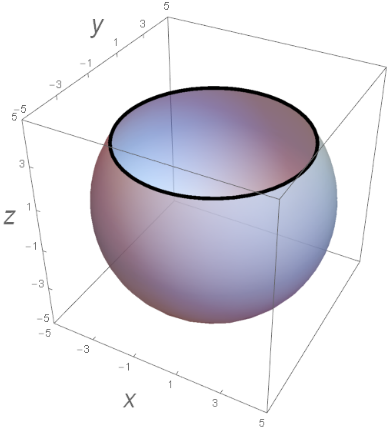
Example: Find the flux of the curl $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the portion of $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward unit normal.



- We use Stokes's Theorem.
- Here, S is the part of $x^2 + y^2 + z^2 = 25$ below $z = 3$. The boundary curve is the intersection of the plane and the sphere.
- The curve has $x^2 + y^2 = 16$ and $z = 3$, which is a circle with parametrization $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3\rangle$ for $0 \le t \le 2\pi$.

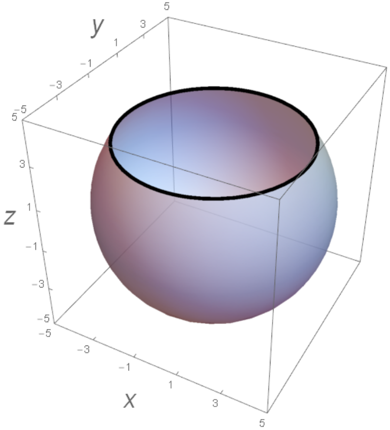
Stokes's Theorem, X

Example: Find the flux of the curl $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the portion of $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward unit normal.



Stokes's Theorem, X

Example: Find the flux of the curl $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the portion of $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward unit normal.



- However, S lies below C, not above it. Since we are using the outward normal, the curve runs clockwise around the surface.
- To use Stokes's Theorem, we need to reverse the orientation of C, which we can do by swapping integration limits: we start at $t = 2\pi$ and end at $t = 0$.

Stokes's Theorem, XI

Example: Find the flux of the curl $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the portion of $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward unit normal.

- From Stokes's Theorem, the flux of the curl is given by the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$.
- Our curve is $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3\rangle$ for $0 \le t \le 2\pi$.

Stokes's Theorem, XI

Example: Find the flux of the curl $\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + e^{x+y}\mathbf{k}$, S is the portion of $x^2 + y^2 + z^2 = 25$ below the plane $z = 3$, and $\mathbf{n}$ is the outward unit normal.

- From Stokes's Theorem, the flux of the curl is given by the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$.
- Our curve is $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 3\rangle$ for $0 \le t \le 2\pi$.
- We have $P = 12\sin(t)$, $Q = -12\cos(t)$, $R = e^{4\cos(t)+4\sin(t)}$, and $dx = -4\sin(t)\,dt$, $dy = 4\cos(t)\,dt$, $dz = 0\,dt$.
- Thus, the desired line integral is $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{0} [(12\sin(t)) \cdot (-4\sin(t)\,dt) + (-12\cos(t)) \cdot (4\cos(t)\,dt)) + e^{4\cos(t)+4\sin(t)} \cdot 0\,dt] = \int_{2\pi}^{0} -48\,dt = 96\pi.$

Normal Form of Green's Theorem

We now discuss the generalization of the normal form of Green's theorem. Here is a reminder of that version of Green's theorem:

Theorem (Green's Theorem, Normal Form)

If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, and $\mathbf{F}$ is a continuously differentiable vector field, then the flux across C is given by

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R (\operatorname{div} \mathbf{F})\, dA$$

where $\mathbf{N}$ is the outward unit normal of the curve.

Gauss's Divergence Theorem, I

And now here is our 3-dimensional generalization of the normal form of Green's theorem:

Theorem (Gauss's Divergence Theorem)

If S is a closed, bounded, piecewise-smooth surface that fully encloses a solid region D, and $\mathbf{F}$ is a continuously differentiable vector field, then the flux across S is given by

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D (\operatorname{div} \mathbf{F}) \, dV$$

where $\mathbf{n}$ is the outward unit normal to the surface.

Gauss's Divergence Theorem, II

In the divergence theorem, the surface S is what we call a "closed" surface, meaning that it encloses a solid region, and does not have a boundary curve: this is unlike in Stokes's theorem, where the surface does not enclose a region.

- To get an idea of the setup, if S is the unit sphere $x^2 + y^2 + z^2 = 1$, then D would be the unit ball $x^2 + y^2 + z^2 \leq 1$.
- If S consists of the 6 faces of the unit cube, then D would be the interior of the cube.
- Inversely, if D is the cylindrical solid $x^2 + y^2 \leq 1$ with $0 \leq z \leq 3$, then S would consist of the top and bottom faces of the cylinder along with the lateral portion of the surface.

Typically, we want to use the Divergence Theorem to convert a surface integral into a triple integral, since the triple integral is usually easier to evaluate.

Gauss's Divergence Theorem, III

Some remarks about the divergence theorem:

- Intuitively, if we think of a vector field as modeling the flow of a fluid, the divergence measures whether there is a "source" or a "sink" at a given point (i.e., whether fluid is flowing inward toward that point, or outward from that point).
- The Divergence Theorem then says that we can measure how much fluid is flowing in or out of a solid region by measuring how much fluid is flowing across its boundary.
- The proof of the Divergence Theorem (which we omit) is essentially the same as the proof of Green's Theorem: we reduce to the case of showing the result for rectangular boxes, parametrize the boxes explicitly, and then glue boxes together to approximate general regions.

Gauss's Divergence Theorem, IV

Example: Find the outward normal flux of the vector field $\mathbf{F}(x, y, z) = \langle x^3 - 3y, 2yz + 1, x^2y^3 \rangle$ through the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

Gauss's Divergence Theorem, IV

Example: Find the outward normal flux of the vector field $\mathbf{F}(x, y, z) = \langle x^3 - 3y, 2yz + 1, x^2y^3 \rangle$ through the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.

- We use the divergence theorem, rather than setting up six separate surface integrals: the flux is $\iiint_V (\mathrm{div}\ \mathbf{F})\ dV$.
- The solid region V is defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and $\mathrm{div}\ \mathbf{F} = 3x^2 + 2z$.
- So by the divergence theorem, the flux is
$$\int_0^1 \int_0^1 \int_0^1 (3x^2 + 2z)\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^1 (3x^2 + 1)\, dy\, dx = \int_0^1 (3x^2 + 1)\, dx = 2.$$

Gauss's Divergence Theorem, V

Example: Evaluate the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ where $\mathbf{F} = \langle x^3 z, y^3 z, x(x^2+y^2)^{3/2} \rangle$ and S is the boundary of the solid cylinder with $4 \le x^2 + y^2 \le 9$ and $0 \le z \le 2$.

Gauss's Divergence Theorem, V

Example: Evaluate the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ where $\mathbf{F} = \langle x^3 z, y^3 z, x(x^2+y^2)^{3/2} \rangle$ and S is the boundary of the solid cylinder with $4 \le x^2 + y^2 \le 9$ and $0 \le z \le 2$.

- We will use the Divergence Theorem. Here, we have $\text{div}(\mathbf{F}) = 3x^2 z + 3y^2 z$.
- Thus the flux is given by $\iiint_D (3x^2 z + 3y^2 z)\, dz\, dy\, dx$, where D is the given solid $4 \le x^2 + y^2 \le 9$ and $0 \le z \le 2$.
- To evaluate this integral we switch to cylindrical coordinates: the region is $0 \le \theta \le 2\pi$, $2 \le r \le 3$, $0 \le z \le 2$, the function is $3r^2 z$, and the differential is $r\, dz\, dr\, d\theta$.
- So the flux is $\int_0^{2\pi} \int_2^3 \int_0^2 3r^2 z \cdot r\, dz\, dr\, d\theta =$

$$\int_0^{2\pi} \int_2^3 6r^3\, dr\, d\theta = \int_0^{2\pi} 195/2\, d\theta = 195\pi.$$

Gauss's Divergence Theorem, VI

Example: Compute the flux $\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = (x^3 + yz)\mathbf{i} + (y^3 + xz)\mathbf{j} + (z^3 + xy)\mathbf{k}$, S is the unit sphere $x^2 + y^2 + z^2 = 1$, and $\mathbf{n}$ is the outward normal.

Gauss's Divergence Theorem, VI

Example: Compute the flux $\oiint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = (x^3 + yz)\mathbf{i} + (y^3 + xz)\mathbf{j} + (z^3 + xy)\mathbf{k}$, S is the unit sphere $x^2 + y^2 + z^2 = 1$, and $\mathbf{n}$ is the outward normal.

- We will use the Divergence Theorem. Here, we have $\text{div}(\mathbf{F}) = 3x^2 + 3y^2 + 3z^2$.
- The region enclosed by S is the unit ball $x^2 + y^2 + z^2 \le 1$.
- Thus the flux is given by $\iiint_D \left(3x^2 + 3y^2 + 3z^2\right) dz\, dy\, dx$, where D is the solid $x^2 + y^2 + z^2 \le 1$.
- To evaluate this integral we switch to spherical coordinates: the region is $0 \le \rho \le 1$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$, the function is $3\rho^2$, and the differential is $\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$.
- So the flux is $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^2 \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{3}{5} \sin(\phi)\, d\phi\, d\theta = \int_0^{2\pi} \frac{6}{5}\, d\theta = \frac{12\pi}{5}.$$

Unified Theory of Integration

As a closing remark, notice that the fundamental theorem of line integrals, Green's theorem, Stokes's theorem, and the divergence theorem collectively unify all of our notions of integration.

- Each of these theorems is a different generalizations of the Fundamental Theorem of Calculus, and they all relate the integral of a function on the boundary of a region to the integral of a derivative on the interior of the region.
- Symbolically, their statements all read as $\int_{\partial R} \omega = \int_R d\omega$, where $d\omega$ represents an appropriate differential of a function ω (e.g., f', ∇f, $\nabla \cdot \mathbf{F}$, or $\nabla \times \mathbf{F}$) and ∂R represents the boundary of the region R.
- The statement above is known as the generalized Stokes theorem, and in fact applies to integration in higher-dimensional spaces as well.

What Now?

Tomorrow's lecture, and Monday's lecture will be devoted to discussing some applications of vector calculus.

- The applications will not appear on the final, but they're a nice way to tie up a lot of the material from the semester (and of course, you will probably see at least some of them in other classes).
- Next Wednesday, the last day of class, I will do some more focused review of exam-style problems.
- I will also schedule a review session during the final exam week (ahead of our exam, of course) where I will go over a number of additional problems from old final exams. The review will be recorded for those who cannot attend live.

Summary

We discussed Stokes's theorem and how it is used.

We described the divergence theorem and how it is used.

Next lecture: Applications of vector calculus, part 1