Math 2321 (Multivariable Calculus)

Lecture #33 of 37 ~ April 8, 2021

Midterm #3 Review #2

Midterm 3 Exam Topics

The topics for the exam are as follows:

- Line integrals
- Surface integrals
- Vector fields
- Work, circulation, and flux integrals in the plane and in 3-space
- Conservative vector fields, potential functions, the fundamental theorem of line integrals
- Divergence and curl of vector fields
- Green's theorem
- Normal and tangential forms of Green's theorem

This represents §4.1 – 4.5 from the notes and WeBWorKs 9-11.

Exam Information

The exam format is the same as the other midterms.

- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- There are approximately 6 pages of material: one page is multiple choice and the rest is free response.
- I have set up a Piazza poll for you to select your desired exam window. Please make your selection by Saturday, April 10th. I will post your selection in Canvas so you can confirm it on Sunday the 11th.
- The "official" exam time limit is $65+25 = 90$ minutes, plus 30 minutes of turnaround time (not to be used for working).

Collaboration of any kind is not allowed. You may not discuss anything about the exam with anyone other than me (the instructor) until 5pm Eastern on Friday, April 16th. This includes Piazza posts.

TRACE Evaluations

The TRACE evaluations for this course will open tomorrow.

- Please do fill out the evaluations. They are quite important (especially for teaching faculty like me) – they are used internally by the department and the university to make decisions about faculty retention, promotion, and course assignments.
- Therefore, if you feel I did a good job teaching this course (and you enjoy things like the review sheets and review sessions, the course notes, etc.), I would ask that you please make the evaluations reflect that.
- Thank you!

And now, on with the lecture....

Review Problems, I

(#1b) Compute $\int_C x\,dx + y\,dy$, where C is the circle $x^2 + y^2 = 9$.

Review Problems, I

(#1b) Compute $\int_C x\,dx + y\,dy$, where C is the circle $x^2 + y^2 = 9$.

- We can parametrize the circle as $\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle$ for $0 \le t \le 2\pi$.
- Then $dx = -3\sin t\,dt$ and $dy = 3\cos t\,dt$.
- Therefore, the integral is

$$\int_0^{2\pi} (3\cos t)(-3\sin t)dt + (3\sin t)(3\cos t)\,dt = \int_0^{2\pi} 0\,dt = \boxed{0}.$$

Review Problems, II

(#6a) Find the divergence and curl of $\mathbf{F} = \langle x^3 + xy,\, y^3 + xy,\, 0 \rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

Review Problems, II

(#6a) Find the divergence and curl of $\mathbf{F} = \left\langle x^3 + xy,\ y^3 + xy,\ 0 \right\rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

- Note $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$ and $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$.
- So $\nabla \cdot \mathbf{F} = \boxed{3x^2 + y + 3y^2 + x}$ and $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, y - x \rangle}$.
- Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, the vector field is $\boxed{\text{not conservative}}$.

Review Problems, III

(#4a) Set up (do not evaluate) an iterated double integral giving the area of the portion of the plane $z = 3x + 4y + 11$ above the region with $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Review Problems, III

(#4a) Set up (do not evaluate) an iterated double integral giving the area of the portion of the plane $z = 3x + 4y + 11$ above the region with $0 \le x \le 1$ and $0 \le y \le 2$.

- We can parametrize this portion of the plane by $\mathbf{r}(s,t) = \langle s, t, 3s + 4t + 11 \rangle$ for $0 \le s \le 1$, $0 \le t \le 2$.
- The function is 1 for surface area.
- Then $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle -3, -4, 1 \rangle$ so $d\sigma = \left|\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|\right| \, ds\, dt = \sqrt{26}\, ds\, dt$.
- Therefore, the surface area integral is $\boxed{\int_0^1 \int_0^2 1 \cdot \sqrt{26}\, ds\, dt}$.

Review Problems, IV

(#9c) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle 2x + 3y, 4x + 5y \rangle$ around the unit circle.

Review Problems, IV

(#9c) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle 2x + 3y, 4x + 5y \rangle$ around the unit circle.

- Since C is a closed curve, we can use Green's theorem to calculate the circulation and the flux:
  Circulation $= \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$ and
  Flux $= \oint_C -Q\,dx + P\,dy = \iint_R (P_x + Q_y)\,dA$.
- Here, the region in polar is $0 \le r \le 1$, $0 \le \theta \le 2\pi$.
- Also, $Q_x - P_y = 1$ and $P_x + Q_y = 7$.
- The circulation is $\int_0^{2\pi} \int_0^1 1 \cdot r\,dr\,d\theta = \boxed{\pi}$.
- The flux is $\int_0^{2\pi} \int_0^1 7 \cdot r\,dr\,d\theta = \boxed{7\pi}$.

Review Problems, V

(#12) Let $\mathbf{F}(x, y) = \langle 1 - y, \cos(y^2) + 2x \rangle$ and let C be the curve starting at $(0, 0)$, traveling along a straight line to $(0, 3)$, then along a straight line to $(4, 3)$, and then along a straight line to $(4, 0)$. Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Review Problems, V

(#12) Let $\mathbf{F}(x, y) = \langle 1 - y, \cos(y^2) + 2x \rangle$ and let C be the curve starting at $(0, 0)$, traveling along a straight line to $(0, 3)$, then along a straight line to $(4, 3)$, and then along a straight line to $(4, 0)$. Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

- This path is not closed because it is missing the segment from $(4, 0)$ to $(0, 0)$ parametrized by $\mathbf{r}(t) = \langle 4 - 4t, 0 \rangle$, $0 \le t \le 1$.
- The line integral on that segment is $\int_C (1 - y)\, dx + (\cos(y^2) + 2x)\, dy =$ $\int_0^1 1 \cdot (-4\, dt) + (9 - 8t) \cdot (0\, dt) = -2$.
- If we add that segment back in, we can use Green's theorem to evaluate the integral along the full path as $\iint_R (Q_x - P_y)\, dA = \int_0^4 \int_0^3 (3)\, dy\, dx = 36$.
- Therefore, the integral on the three requested pieces is equal to the difference $36 - (-2) = \boxed{38}$.

Review Problems, VI

(#1a) Compute $\int_C x^2\, ds$, where C is the line segment from $(0, 1)$ to $(3, 2)$.

Review Problems, VI

(#1a) Compute $\int_C x^2\,ds$, where C is the line segment from $(0,1)$ to $(3,2)$.

- We can parametrize the line segment as $\mathbf{r}(t) = \langle 3t, 1+t\rangle$ for $0 \le t \le 1$.
- Then $||\mathbf{v}|| = ||\langle 3,1\rangle|| = \sqrt{10}$, so the differential is $ds = ||\mathbf{v}(t)||\ dt = \sqrt{10}\,dt$.
- The function is $x^2 = (3t)^2 = 9t^2$.
- The line integral is then $\displaystyle\int_0^1 (3t)^2\sqrt{10}\,dt = \boxed{3\sqrt{10}}$.

Review Problems, VI

(#2b) Find an equation for the tangent plane to the surface parametrized by $\mathbf{r}(s,t) = \left\langle s^2, 2st, 3t^3 \right\rangle$ at the point $(x, y, z) = (1, 2, -3)$.

Review Problems, VI

(#2b) Find an equation for the tangent plane to the surface parametrized by $\mathbf{r}(s,t) = \langle s^2, 2st, 3t^3 \rangle$ at the point $(x,y,z) = (1,2,-3)$.

- If $(x,y,z) = (1,2,-3)$ then $3t^3 = -3$ requires $t = -1$ and then $2st = 2$ requires also $s = -1$. Then $s^2 = 1$ as also required.
- Then $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 2s, 2t, 0 \rangle \times \langle 0, 2s, 9t^2 \rangle$, so $s = -1$ and $t = -1$ give the normal vector $\mathbf{n} = \langle -2, -2, 0 \rangle \times \langle 0, -2, 9 \rangle = \langle -18, 18, 4 \rangle$.
- Then the equation for the tangent plane is $\boxed{-18(x-1) + 18(y-2) + 4(z+3) = 0}$.

Review Problems, VII

(#3d) Let S be the unit sphere. Parametrize S and then set up (do not evaluate) the integral $\iint_S z^2 \, d\sigma$.

Review Problems, VII

(#3d) Let S be the unit sphere. Parametrize S and then set up (do not evaluate) the integral $\iint_S z^2 \, d\sigma$.

- Using spherical coordinates, we can parametrize the unit sphere $\rho = 1$ by $\mathbf{r}(\theta, \varphi) = \langle \cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi \rangle$ for $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$.
- The function is then $z^2 = \cos^2\varphi$.
- Then $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \left\langle -\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin\varphi \cos\varphi \right\rangle$ and so $d\sigma = \left|\left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right|\right| \, d\varphi \, d\theta = \sin\varphi \, d\varphi \, d\theta$.
- Thus, the surface integral is $\boxed{\int_0^{2\pi} \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi \, d\theta}$.

Review Problems, VIII

(#5d) Compute the outward normal flux of $\mathbf{F} = \langle y, -x, z \rangle$ across the portion of the surface $z = x^2 + y^2$ below $z = 2$, with upward orientation.

Review Problems, VIII

(#5d) Compute the outward normal flux of $\mathbf{F} = \langle y, -x, z\rangle$ across the portion of the surface $z = x^2 + y^2$ below $z = 2$, with upward orientation.

- In cylindrical S is $z = r^2$ so with parameters r, θ we get $\mathbf{r}(r, \theta) = \left\langle r\cos\theta, r\sin\theta, r^2\right\rangle$ for $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{2}$.
- Then $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r\right\rangle$, which has upward orientation since z is positive.
- Since $\mathbf{F} = \left\langle r\sin\theta, -r\cos\theta, r^2\right\rangle$, we see $\mathbf{F}\cdot\mathbf{n} = \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r\right\rangle \cdot \left\langle r\sin\theta, -r\cos\theta, r^2\right\rangle = r^3$.
- Thus, the flux is $\int_0^{2\pi}\int_0^{\sqrt{2}} r^3\,dr\,d\theta = \int_0^{2\pi} 1\,d\theta = \boxed{2\pi}$.

Review Problems, VII

(#5e) Compute the outward normal flux of $\mathbf{F} = \langle x, y, z \rangle$ across the upper half of the unit sphere, with outward orientation.

Review Problems, VII

(#5e) Compute the outward normal flux of $\mathbf{F} = \langle x, y, z \rangle$ across the upper half of the unit sphere, with outward orientation.

- In spherical coordinates the surface S is $\rho = 1$, so we get a parametrization $\mathbf{r}(\theta, \varphi) = \langle \cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi \rangle$ for $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi/2$.
- Then $\frac{\partial \mathbf{r}}{\partial \varphi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \left\langle \cos\theta \sin^2\varphi, \sin\theta \sin^2\varphi, \sin\varphi \cos\varphi \right\rangle$, which has outward orientation since the signs are positive.
- Since $\mathbf{F} = \langle \cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi \rangle$, we see $\mathbf{F} \cdot \mathbf{n} = \left\langle \cos\theta \sin^2\varphi, \sin\theta \sin^2\varphi, \sin\varphi \cos\varphi \right\rangle \cdot \langle \cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi \rangle$ $= \cos^2\theta \sin^3\varphi + \sin^2\theta \sin^3\varphi + \sin\varphi \cos^2\varphi = \sin\varphi$.
- Thus, the flux is $\int_0^{2\pi} \int_0^{\pi/2} \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} 1 \, d\theta = \boxed{2\pi}$.

Review Problems, VIII

(#4d) Set up (do not evaluate) an iterated double integral giving the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ that lies below the cone $z = \sqrt{3} \cdot \sqrt{x^2 + y^2}$.

Review Problems, VIII

(#4d) Set up (do not evaluate) an iterated double integral giving the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ that lies below the cone $z = \sqrt{3} \cdot \sqrt{x^2 + y^2}$.

- In spherical, the cone is $\varphi = \pi/6$, so we can parametrize S by $\mathbf{r}(\theta, \varphi) = \langle 3\cos\theta\sin\varphi, 3\sin\theta\sin\varphi, 3\cos\varphi\rangle$ for $0 \le \theta \le 2\pi$, $\pi/6 \le \varphi \le \pi$.
- The function is 1 for surface area. Then $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \left\langle -9\cos\theta\sin^2\varphi, -9\sin\theta\sin^2\varphi, -9\sin\varphi\cos\varphi \right\rangle$ and so $d\sigma = \left|\left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi}\right|\right| \, d\varphi\, d\theta = 9\sin\varphi\, d\varphi\, d\theta$.
- Thus, the surface area is $\boxed{\int_0^{2\pi} \int_{\pi/6}^{\pi} 1 \cdot 9\sin\varphi\, d\varphi\, d\theta}$.

Review Problems, IX

(#7) Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$, where C is the curve $\mathbf{r}(t) = \langle te^t, \arctan(t), \ln(1+t)\rangle$ for $0 \le t \le 1$.

Review Problems, IX

(#7) Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$, where C is the curve $\mathbf{r}(t) = \langle te^t, \arctan(t), \ln(1+t)\rangle$ for $0 \le t \le 1$.

- This integral represents the work done by $\mathbf{F} = \langle yz, xz, xy\rangle$ along the curve C from $\mathbf{r}(0) = \langle 0, 0, 0\rangle$ to $\mathbf{r}(1) = \langle e, \pi/4, \ln(2)\rangle$.
- We could set up the line integral but it is messy.
- Instead, we can skip all of that calculation by noting $\mathbf{F}$ is conservative with potential $U = xyz$.
- So by the fundamental theorem of line integrals, the work is simply $U(e, \pi/4, \ln(2)) - U(0, 0, 0) = \boxed{e \cdot (\pi/4) \cdot \ln(2)}$.

Review Problems, X

(#6d) Find the divergence and curl of $\mathbf{F} = \left\langle 2xyz + e^z, \, x^2z + 3, \, xe^z + x^2y + 2 \right\rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

Review Problems, X

(#6d) Find the divergence and curl of $\mathbf{F} = \left\langle 2xyz + e^z,\ x^2z + 3,\ xe^z + x^2y + 2 \right\rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

- Note $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$ and $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$.
- So $\nabla \cdot \mathbf{F} = \boxed{2yz + xe^z}$, $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, 0 \rangle}$.
- Since $\nabla \times \mathbf{F} = \mathbf{0}$, field is $\boxed{\text{conservative}}$.
- The potential has $U_x = 2xyz + e^z$, $U_y = x^2z + 3$, and $U_z = xe^z + x^2y + 2$.
- So we can take $U = \boxed{x^2yz + xe^z + 3y + 2z}$.

Review Problems, XI

(#11) A particle moves through the force field $\mathbf{F}(x, y) = \left\langle 9y - e^{x^2} + 11, 4x + \sin\sqrt{y} \right\rangle$ N where x and y are measured in meters. Decide if $\mathbf{F}$ is conservative, then calculate the work done by $\mathbf{F}$ if the particle starts at $(0, 0)$, moves along a straight line to $(0, 4)$, goes counterclockwise along the circle $x^2 + y^2 = 16$ to $(-4, 0)$, then goes in a straight line back to $(0, 0)$.

Review Problems, XI

(#11) A particle moves through the force field $\mathbf{F}(x,y) = \left\langle 9y - e^{x^2} + 11,\ 4x + \sin\sqrt{y} \right\rangle$ N where x and y are measured in meters. Decide if $\mathbf{F}$ is conservative, then calculate the work done by $\mathbf{F}$ if the particle starts at $(0,0)$, moves along a straight line to $(0,4)$, goes counterclockwise along the circle $x^2 + y^2 = 16$ to $(-4,0)$, then goes in a straight line back to $(0,0)$.

- We compute $\mathrm{curl}(\mathbf{F}) = \langle 0, 0, -5 \rangle$. This is nonzero so $\mathbf{F}$ is $\boxed{\text{not conservative}}$.
- For the work we can use Green's theorem since the path is the counterclockwise boundary of the polar region $0 \le r \le 4$ and $\pi/2 \le \theta \le \pi$. By Green, the work is $\iint_R (Q_x - P_y)\, dA = \int_{\pi/2}^{\pi} \int_0^4 -5 \cdot r\, dr\, d\theta = \boxed{-20\pi \text{ J}}$.

Review Problems, XII

(#10a) Let $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(-2, 0)$ to $(0, 0)$, C_2 is the line segment from $(0, 0)$ to $(\sqrt{2}, \sqrt{2})$, and C_3 is the shorter arc of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(-2, 0)$. Find the outward flux of
$\mathbf{F} = \left\langle 2x^3y^2 + y^3,\, x^2 - 2x^2y^3 \right\rangle$ around C.

Review Problems, XII

(#10a) Let $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(-2,0)$ to $(0,0)$, C_2 is the line segment from $(0,0)$ to $(\sqrt{2},\sqrt{2})$, and C_3 is the shorter arc of the circle $x^2+y^2=4$ from $(\sqrt{2},\sqrt{2})$ to $(-2,0)$. Find the outward flux of $\mathbf{F} = \langle 2x^3y^2+y^3,\, x^2-2x^2y^3\rangle$ around C.

- Note that C is the counterclockwise boundary of the polar region $0 \le r \le 2$ and $\pi/4 \le \theta \le \pi$.
- Since C is a closed curve, we can use Green's theorem to get the flux: Flux $= \oint_C -Q\,dx + P\,dy = \iint_R (P_x+Q_y)\,dA$.
- We calculate $P_x + Q_y = 6x^2y^2 - 6x^2y^2 = 0$.
- The flux is therefore
$$\iint_R (P_x+Q_y)\,dA = \int_{\pi/4}^{\pi}\int_0^2 0\cdot r\,dr\,d\theta = \boxed{0}.$$

Review Problems, XIII

(#10b) Let $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(-2, 0)$ to $(0, 0)$, C_2 is the line segment from $(0, 0)$ to $(\sqrt{2}, \sqrt{2})$, and C_3 is the shorter arc of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(-2, 0)$. Find the counterclockwise circulation of $\mathbf{F} = \langle x^2 - y^3, x^3 + y^4 \rangle$ around C.

Review Problems, XIII

(#10b) Let $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(-2, 0)$ to $(0, 0)$, C_2 is the line segment from $(0, 0)$ to $(\sqrt{2}, \sqrt{2})$, and C_3 is the shorter arc of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(-2, 0)$. Find the counterclockwise circulation of $\mathbf{F} = \langle x^2 - y^3, x^3 + y^4 \rangle$ around C.

- Note that C is the counterclockwise boundary of the polar region $0 \le r \le 2$ and $\pi/4 \le \theta \le \pi$.
- Since C is a closed curve, we can use Green's theorem to get the circulation:
  Circulation $= \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$.
- We calculate $Q_x - P_y = 3x^2 + 3y^2 = 3r^2$.
- The circulation is therefore
  $$\iint_R (Q_x - P_y)\,dA = \int_{\pi/4}^{\pi} \int_0^2 3r^2 \cdot r\,dr\,d\theta = \boxed{9\pi}.$$

Review Problems, XIV

(#10c) Let $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(-2, 0)$ to $(0, 0)$, C_2 is the line segment from $(0, 0)$ to $(\sqrt{2}, \sqrt{2})$, and C_3 is the shorter arc of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(-2, 0)$. Find the work done by $\mathbf{F} = \langle -2y, 2x \rangle$ on a particle that travels once around C.

Review Problems, XIV

(#10c) Let $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line segment from $(-2, 0)$ to $(0, 0)$, C_2 is the line segment from $(0, 0)$ to $(\sqrt{2}, \sqrt{2})$, and C_3 is the shorter arc of the circle $x^2 + y^2 = 4$ from $(\sqrt{2}, \sqrt{2})$ to $(-2, 0)$. Find the work done by $\mathbf{F} = \langle -2y, 2x \rangle$ on a particle that travels once around C.

- Note that C is the counterclockwise boundary of the polar region $0 \le r \le 2$ and $\pi/4 \le \theta \le \pi$.
- We can use the tangential form of Green's theorem for the work integral, since it is the same as the circulation integral: Work $= \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$.
- We calculate $Q_x - P_y = 4$.
- The work is therefore $\displaystyle\int_{\pi/4}^{\pi} \int_0^2 4 \cdot r\,dr\,d\theta = \boxed{6\pi}$.

Review Problems, XV

(#5c) Compute the outward normal flux of $\mathbf{F} = \langle xz, yz, z^4 \rangle$ across the portion of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$, with outward orientation.

Review Problems, XV

(#5c) Compute the outward normal flux of $\mathbf{F} = \left\langle xz,\ yz,\ z^4 \right\rangle$ across the portion of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$, with outward orientation.

- In cylindrical S is $r = 2$ so with parameters θ, z we get $\mathbf{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$ for $0 \le \theta \le 2\pi$, $0 \le z \le 3$.
- Then $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$, which has outward orientation since x, y are positive.
- Since $\mathbf{F} = \left\langle 2z\cos\theta, 2z\sin\theta, z^4 \right\rangle$, we see $\mathbf{F} \cdot \mathbf{n} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle \cdot \left\langle 2z\cos\theta, 2z\sin\theta, z^4 \right\rangle = 4z$.
- Thus, the flux is $\int_0^{2\pi} \int_0^3 4z\, dz\, d\theta = \int_0^{2\pi} 18\, d\theta = \boxed{36\pi}$.

Review Problems, XVI

(#3a) Let S be the portion of the plane $z = 2x + 2y$ above the rectangle with $1 \leq x \leq 2$, $2 \leq y \leq 4$. Parametrize S and then set up (do not evaluate) the integral $\iint_S (x^2 + y^2)\, d\sigma$.

Review Problems, XVI

(#3a) Let S be the portion of the plane $z = 2x + 2y$ above the rectangle with $1 \le x \le 2$, $2 \le y \le 4$. Parametrize S and then set up (do not evaluate) the integral $\iint_S (x^2 + y^2)\, d\sigma$.

- Using rectangular coordinates we can parametrize S by $\mathbf{r}(s,t) = \langle s, t, 2s + 2t \rangle$ for $1 \le s \le 2$, $2 \le t \le 4$.
- Then the function is $x^2 + y^2 = s^2 + t^2$.
- For the differential, $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle -2, -2, 1 \rangle$ so $d\sigma = \left|\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|\right| \, ds\, dt = 3\, ds\, dt$.
- Thus, the surface integral is $\boxed{\int_2^4 \int_1^2 (s^2 + t^2) \cdot 3\, ds\, dt}$.

Review Problems, XVII

(#1g) Compute the (outward normal) flux of $\mathbf{F}(x, y) = \left\langle xy,\ y^2 \right\rangle$ across, and the circulation of $\mathbf{F}$ along, the curve $\mathbf{r}(t) = \left\langle t^2, t^3 \right\rangle$ for $0 \le t \le 1$.

Review Problems, XVII

(#1g) Compute the (outward normal) flux of $\mathbf{F}(x, y) = \langle xy,\, y^2 \rangle$ across, and the circulation of $\mathbf{F}$ along, the curve $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ for $0 \le t \le 1$.

- With $x = t^2$, $y = t^3$ we get $dx = 2t\,dt$, $dy = 3t^2\,dt$, $P = xy = t^5$, $Q = y^2 = t^6$.
- Circulation is $\int_C P\,dx + Q\,dy = \int_0^1 t^5(2t\,dt) + t^6(3t^2\,dt) = \int_0^1 (2t^6 + 3t^8)\,dt = \boxed{13/21}$.
- Flux is $\int_C -Q\,dx + P\,dy = \int_0^1 -t^6(2t\,dt) + t^5(3t^2\,dt) = \int_0^1 t^7\,dt = \boxed{1/8}$.

Review Problems, XVIII

(#3c) Let S be the portion of the cone $z = 4\sqrt{x^2 + y^2}$ below $z = 8$. Parametrize S and then set up (do not evaluate) the integral $\iint_S \sqrt{x^2 + y^2}\, d\sigma$.

Review Problems, XVIII

(#3c) Let S be the portion of the cone $z = 4\sqrt{x^2 + y^2}$ below $z = 8$. Parametrize S and then set up (do not evaluate) the integral $\iint_S \sqrt{x^2 + y^2}\, d\sigma$.

- In cylindrical coordinates the surface is $z = 4r$, so we use parameters r, θ. The parametrization of this portion is $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 4r\rangle$ for $0 \le \theta \le 2\pi$, $0 \le r \le 2$.
- The function is $x^2 + y^2 = r^2$.
- For the differential, we compute $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -4r\cos\theta,\ -4r\sin\theta, r\rangle$ so $d\sigma = \left|\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right|\right|\, dr\, d\theta = r\sqrt{17}\, dr\, d\theta$.
- Thus, the surface integral is $\boxed{\int_0^{2\pi}\int_0^2 r \cdot r\sqrt{17}\, dr\, d\theta}$.

Review Problems, XIX

(#5a) Compute the outward normal flux of $\mathbf{F} = \langle x^2, 2x^2, 3x^2 \rangle$ across the portion of the plane $x + y + z = 4$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$ having upward orientation.

Review Problems, XIX

(#5a) Compute the outward normal flux of $\mathbf{F} = \left\langle x^2,\, 2x^2,\, 3x^2 \right\rangle$ across the portion of the plane $x + y + z = 4$ with $0 \le x \le 1$ and $0 \le y \le 2$ having upward orientation.

- We can parametrize this part of the plane as $\mathbf{r}(s,t) = \langle s, t, 4 - s - t \rangle$ for $0 \le s \le 1$, $0 \le t \le 2$.
- Then $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle 1, 1, 1 \rangle$. This has upward orientation since the z-coordinate is positive.
- Since $\mathbf{F} = \left\langle s^2, 2s^2, 3s^2 \right\rangle$, we see $\mathbf{F} \cdot \mathbf{n} = 6s^2$.
- Thus, the flux is $\int_0^2 \int_0^1 6s^2 \, ds \, dt = \boxed{4}$.

Review Problems, XX

(#1c) Compute the integral of $\sqrt{x^2 + y^2}$ on the portion of the helix $\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$ for $0 \le t \le \pi$.

Review Problems, XX

(#1c) Compute the integral of $\sqrt{x^2+y^2}$ on the portion of the helix $\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$ for $0 \le t \le \pi$.

- We are given the parametrization already.
- Here, we have $\mathbf{v}(t) = \langle -3\sin 3t, 3\cos 3t, 4 \rangle$ and so $ds = ||\mathbf{v}(t)|| = ||\langle -3\sin 3t,\ 3\cos 3t, 4 \rangle|| = 5$.
- The function is $\sqrt{x^2+y^2} = 1$.
- Thus, the line integral is $\int_0^\pi 1 \cdot 5\,dt = \boxed{5\pi}$.

Review Problems, XXI

(#4b) Set up (do not evaluate) an iterated double integral giving the area of the portion of the surface parametrized by $\mathbf{r}(s,t) = \left\langle s^2, st, t^2 \right\rangle$ with $0 \le s \le 1$ and $0 \le t \le 2$.

Review Problems, XXI

(#4b) Set up (do not evaluate) an iterated double integral giving the area of the portion of the surface parametrized by $\mathbf{r}(s,t) = \left\langle s^2, st, t^2 \right\rangle$ with $0 \le s \le 1$ and $0 \le t \le 2$.

- The function is 1 for surface area.
- Here $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle 2s, t, 0 \rangle \times \langle 0, s, 2t \rangle = \left\langle 2t^2, -4st, 2s^2 \right\rangle$ so $d\sigma = \left|\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|\right| \, ds \, dt = 2\sqrt{s^4 + 4s^2t^2 + t^4} \, ds \, dt$.
- Thus, the surface area integral is $\boxed{\int_0^1 \int_0^2 1 \cdot 2\sqrt{s^4 + 4s^2t^2 + t^4} \, ds \, dt}$.
- Sadly, this does not really simplify in any nice way.

Review Problems, XXII

(#1d) Compute the average value of y on the upper half of the circle $x^2 + y^2 = 4$.

Review Problems, XXII

(#1d) Compute the average value of y on the upper half of the circle $x^2 + y^2 = 4$.

- The average value is $\left[\int_C y\,ds\right] / \left[\int_C 1\,ds\right]$.
- We can parametrize the curve as $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t\rangle$ for $0 \le t \le \pi$, so $\mathbf{v}(t) = \langle -2\sin t, 2\cos t\rangle$.
- Then $ds = ||\mathbf{v}(t)||\ dt = 2\,dt$.
- The numerator integral is $\int_0^\pi 2\sin t \cdot 2\,dt = 8$, while the denominator integral is $\int_0^\pi 2\,dt = 2\pi$.
- Hence the average value is $\boxed{8/(2\pi) = 4/\pi}$.

Summary

We did some more review problems for midterm 2.

Next lecture: Stokes's Theorem, the Divergence Theorem.

Note that there are no classes on Monday April 12th, so our next lecture is after the midterm, on Wednesday April 14th.