Math 2321 (Multivariable Calculus)

Lecture #32 of 37 $\sim$ April 7, 2021

Midterm #3 Review #1

Midterm 3 Exam Topics

The topics for the exam are as follows:

- Line integrals
- Surface integrals
- Vector fields
- Work, circulation, and flux integrals in the plane and in 3-space
- Conservative vector fields, potential functions, the fundamental theorem of line integrals
- Divergence and curl of vector fields
- Green's theorem
- Normal and tangential forms of Green's theorem

This represents §4.1 – 4.5 from the notes and WeBWorKs 9-11.

Exam Information

The exam format is the same as the other midterms.

- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- There are approximately 6 pages of material: one page is multiple choice and the rest is free response.
- I have set up a Piazza poll for you to select your desired exam window. Please make your selection by Saturday, April 10th. I will post your selection in Canvas so you can confirm it on Sunday the 11th.
- The "official" exam time limit is $65+25 = 90$ minutes, plus 30 minutes of turnaround time (not to be used for working).

Collaboration of any kind is not allowed. You may not discuss anything about the exam with anyone other than me (the instructor) until 5pm Eastern on Friday, April 16th. This includes Piazza posts.

Review Problems, I

(#1e) Compute the work done by $\mathbf{F}(x, y, z) = \langle yz,\ xz,\ x^3 \rangle$ N on a particle that travels from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve parametrized by $\mathbf{r}(t) = \langle t^2,\ t^4,\ t^3 \rangle$ m.

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- The work is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$.
- With $x = t^2$, $y = t^4$, $z = t^3$ we get $dx = 2t\,dt$, $dy = 4t^3\,dt$, $dz = 3t^2\,dt$, and also $P = yz = t^7$, $Q = xz = t^5$, $R = x^3 = t^6$.
- Thus, the work integral is $\int_0^1 t^7 \cdot 2t\,dt + t^5 \cdot 4t^3\,dt + t^6 \cdot 3t^2\,dt = \int_0^1 9t^8\,dt = 1$. The correct units here are joules, so the answer is $\boxed{1\text{ J}}$.

Review Problems, II

(#2a) Find an equation for the tangent plane to the surface parametrized by $\mathbf{r}(s,t) = \left\langle s+t, s^2+t^2, s^3+t^3 \right\rangle$ where $s = 1$ and $t = 2$.

Review Problems, II

(#2a) Find an equation for the tangent plane to the surface parametrized by $\mathbf{r}(s,t) = \langle s+t, s^2+t^2, s^3+t^3 \rangle$ where $s = 1$ and $t = 2$.

- The normal vector to the tangent plane is $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt)$

Review Problems, II

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- The normal vector to the tangent plane is $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \left\langle 1, 2s, 3s^2 \right\rangle \times \left\langle 1, 2t, 3t^2 \right\rangle$.
- With $s = 1$ and $t = 2$ this is $\mathbf{n} = \langle 1, 2, 3 \rangle \times \langle 1, 4, 12 \rangle = \langle 12, -9, 2 \rangle$.
- The tangency point is $\mathbf{r}(1,2) = \langle 3, 5, 9 \rangle$.
- Thus, the tangent plane's equation is $\boxed{12(x-3) - 9(y-5) + 2(z-9) = 0}$.

Review Problems, III

(#5b) Compute the flux of $\mathbf{F} = \left\langle -xz, -yz, x^2 + y^2 \right\rangle$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 6$, with upward orientation.

Review Problems, III

(#5b) Compute the flux of $\mathbf{F} = \left\langle -xz, -yz, x^2 + y^2 \right\rangle$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 6$, with upward orientation.

- In cylindrical the cone is $z = r$, so using parameters r, θ we get the parametrization $\mathbf{r}(r, \theta) = \langle r \cos\theta, r \sin\theta, r \rangle$ for $0 \leq r \leq \sqrt{6}$, $0 \leq \theta \leq 2\pi$.

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- Then $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r\cos\theta, -r\sin\theta, r\rangle$, which has upward orientation since the z-coordinate is positive.
- Then $\mathbf{F} = \left\langle -r^2\cos\theta, -r^2\sin\theta, r^2\right\rangle$ and so $\mathbf{F} \cdot \mathbf{n} = \left\langle -r^2\cos\theta, -r^2\sin\theta, r^2\right\rangle \cdot \langle -r\cos\theta, -r\sin\theta, r\rangle = 2r^3$.
- Thus, the flux is $\int_0^{2\pi}\int_0^{\sqrt{6}} 2r^3\,dr\,d\theta = \int_0^{2\pi} 18\,d\theta = \boxed{36\pi}$.

Review Problems, IV

(#9a) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle xy^2, \, y^3 \rangle$ around the boundary of the rectangle with vertices $(0,0)$, $(2,0)$, $(2,3)$, $(0,3)$.

Review Problems, IV

(#9a) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle xy^2, y^3 \rangle$ around the boundary of the rectangle with vertices $(0,0)$, $(2,0)$, $(2,3)$, $(0,3)$.

- Since C is a closed curve, we can use Green's theorem to calculate the circulation and the flux:
  Circulation $= \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$ and
  Flux $= \oint_C -Q\,dx + P\,dy = \iint_R (P_x + Q_y)\,dA$.
- Here, the region is $0 \le x \le 2$, $0 \le y \le 3$.
- Also, $Q_x - P_y = -2xy$ and $P_x + Q_y = 4y^2$.
- The circulation is $\int_0^2 \int_0^3 (-2xy)\,dy\,dx = \boxed{-18}$.
- The flux is $\int_0^2 \int_0^3 4y^2\,dy\,dx = \boxed{72}$.

Review Problems, V

(#4c) Set up (do not evaluate) an iterated double integral giving the area of the portion of the cone $z = 3\sqrt{x^2 + y^2}$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

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- In cylindrical coordinates, the equation is $z = 3r$, so we use parameters r, θ. The parametrization of this portion is $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle$ for $0 \le \theta \le 2\pi$, $2 \le r \le 3$.

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- In cylindrical coordinates, the equation is $z = 3r$, so we use parameters r, θ. The parametrization of this portion is $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle$ for $0 \le \theta \le 2\pi$, $2 \le r \le 3$.
- The function is 1 for surface area.
- For the differential, we compute $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -3r\cos\theta,\ -3r\sin\theta, r\rangle$ so $d\sigma = \left|\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right|\right|\, dr\, d\theta = r\sqrt{10}\, dr\, d\theta$.
- Thus, the surface area integral is $\boxed{\int_0^{2\pi}\int_2^3 1 \cdot r\sqrt{10}\, dr\, d\theta} = 5\pi\sqrt{10}$.

Review Problems, VI

(#6b) Find the divergence and curl of $\mathbf{F} = \langle yz + 2x, xz + 2z, xy + 2y \rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

Review Problems, VI

(#6b) Find the divergence and curl of $\mathbf{F} = \langle yz + 2x, xz + 2z, xy + 2y \rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

- Note $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$ and $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.
- So $\nabla \cdot \mathbf{F} = \boxed{2}$ and $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, 0 \rangle}$.

Review Problems, VI

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- So $\nabla \cdot \mathbf{F} = \boxed{2}$ and $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, 0 \rangle}$.
- Since $\nabla \times \mathbf{F} = \mathbf{0}$ and there are no holes in the domain, this means the vector field is $\boxed{\text{conservative}}$.
- The potential has $U_x = yz + 2x$, $U_y = xz + 2z$, $U_z = xy + 2y$, so we can take $U = \boxed{xyz + x^2 + 2yz}$.

Review Problems, VII

(#8) Let C be the curve that runs once counterclockwise around the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Find $\oint_C (x^2 + y)\,dx + (2xy^2 - xy)\,dy$.

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- We could set up the four line integrals and evaluate them all. However, this is much more work than necessary, because we can use Green's theorem instead.
- By Green's theorem, the integral $\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dy\,dx$, where R is the region enclosed by C, which here is the square $0 \le x \le 1$, $0 \le y \le 1$.

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- We could set up the four line integrals and evaluate them all. However, this is much more work than necessary, because we can use Green's theorem instead.
- By Green's theorem, the integral $\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dy\,dx$, where R is the region enclosed by C, which here is the square $0 \le x \le 1$, $0 \le y \le 1$.
- Since $Q_x - P_y = (2y^2 - y) - (1)$, our integral equals $\int_0^1 \int_0^1 (2y^2 - y - 1)\,dy\,dx = \boxed{-5/6}$.

Review Problems, VIII

(#1f) Compute the (outward normal) flux of $\mathbf{F}(x, y) = \langle y + x,\, y - x \rangle$ across, and the circulation of $\mathbf{F}$ along, the path C along the upper half of $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.

Review Problems, VIII

(#1f) Compute the (outward normal) flux of $\mathbf{F}(x, y) = \langle y + x,\ y - x\rangle$ across, and the circulation of $\mathbf{F}$ along, the path C along the upper half of $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.

- We can parametrize this curve as $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \leq t \leq \pi$. Then $x = \cos t$, $y = \sin t$.
- Thus $dx = -\sin t\, dt$, $dy = \cos t\, dt$, and also $P = y + x = \sin t + \cos t$, $Q = y - x = \sin t - \cos t$.
- Circulation is $\int_C P\, dx + Q\, dy = \int_0^\pi (\sin t + \cos t)(-\sin t\, dt) + (\sin t - \cos t)(\cos t\, dt) = \int_0^\pi -1\, dt = \boxed{-\pi}$.
- Flux is $\int_C -Q\, dx + P\, dy = \int_0^\pi (\cos t - \sin t)(-\sin t\, dt) + (\sin t + \cos t)(\cos t\, dt) = \int_0^\pi 1\, dt = \boxed{\pi}$.

Review Problems, IX

(#9d) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle x^3 - y^3, x^3 + y^3 \rangle$ around the boundary of the quarter-disc $x^2 + y^2 \leq 16$ in the first quadrant.

Review Problems, IX

(#9d) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle x^3 - y^3, x^3 + y^3 \rangle$ around the boundary of the quarter-disc $x^2 + y^2 \leq 16$ in the first quadrant.

- Since C is a closed curve, we can use Green's theorem to calculate the circulation and the flux:
  Circulation $= \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$ and
  Flux $= \oint_C -Q\,dx + P\,dy = \iint_R (P_x + Q_y)\,dA$.
- Here, the region in polar is $0 \leq r \leq 4$, $0 \leq \theta \leq \pi/2$.
- Also, $Q_x - P_y = 3x^2 + 3y^2 = 3r^2$ and
  $P_x + Q_y = 3x^2 + 3y^2 = 3r^2$.
- The circulation is $\int_0^{\pi/2} \int_0^4 r^2 \cdot r\,dr\,d\theta = \boxed{32\pi}$.
- The flux is $\int_0^{\pi/2} \int_0^4 r^2 \cdot r\,dr\,d\theta = \boxed{32\pi}$.

Review Problems, X

(#5f) Compute the outward normal flux of $\mathbf{F} = \langle y^3, x^3, 1 \rangle$ across the portion of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, with upward orientation.

(#5f) Compute the outward normal flux of $\mathbf{F} = \langle y^3, x^3, 1 \rangle$ across the portion of the paraboloid $z = 1 - x^2 - y^2$ lying above the xy-plane, with upward orientation.

- In cylindrical coordinates S is $z = 1 - r^2$, so using parameters r, θ yields $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 1 - r^2 \rangle$ for $0 \le \theta \le 2\pi$, $0 \le r \le 1$.
- Then $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$ which has upward orientation since the z-coordinate is positive.
- Since $\mathbf{F} = \langle r^3\sin^3\theta, r^3\cos^3\theta, 1 \rangle$, we see $\mathbf{F} \cdot \mathbf{n} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle \cdot \langle r^3\sin^3\theta, r^3\cos^3\theta, 1 \rangle = 2r^5\sin\theta\cos\theta + r$.
- Thus, the flux is $\int_0^{2\pi}\int_0^1 (2r^5\sin\theta\cos\theta + r)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{3}\sin\theta\cos\theta + \frac{1}{3}\right)d\theta = \boxed{\pi}$.

Review Problems, XI

(#3b) Let S be the portion of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 4$. Parametrize S and then set up (do not evaluate) the integral $\iint_S (zx^2 + zy^2)\, d\sigma$.

Review Problems, XI

(#3b) Let S be the portion of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 4$. Parametrize S and then set up (do not evaluate) the integral $\iint_S (zx^2 + zy^2)\, d\sigma$.

- In cylindrical coordinates the surface is $r = 2$, so we use z and θ as the parameters. The parametrization of this portion is $\mathbf{r}(z, \theta) = \langle 2\cos\theta,\ 2\sin\theta,\ z\rangle$ for $0 \le \theta \le 2\pi$, $0 \le z \le 4$.
- The function is $zx^2 + zy^2 = 4z$.
- For the differential, we compute $\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -2\cos\theta, 2\sin\theta, 0\rangle$, and therefore $d\sigma = \left|\left|\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \theta}\right|\right|\, dz\, d\theta = 2\, dz\, d\theta$.
- So, the surface integral is $\boxed{\int_0^{2\pi} \int_0^4 4z \cdot 2\, dz\, d\theta}$.

Review Problems, XII

(#9b) Find the counterclockwise circulation and the outward normal flux of $\mathbf{F} = \langle 6xy,\, 0 \rangle$ around the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$

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  Circulation $= \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$ and
  Flux $= \oint_C -Q\,dx + P\,dy = \iint_R (P_x + Q_y)\,dA$.
- Here, the region is $0 \le x \le 1$, $0 \le y \le 2 - 2x$.
- Also, $Q_x - P_y = -6x$ and $P_x + Q_y = 6y$.
- The circulation is $\int_0^1 \int_0^{2-2x} -6x\,dy\,dx = \boxed{-2}$.
- The flux is $\int_0^1 \int_0^{2-2x} 6y\,dy\,dx = \boxed{4}$.

Review Problems, XIII

(#6c) Find the divergence and curl of $\mathbf{F} = \langle x^2yz, x^2z^2, 2x^2yz \rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

Review Problems, XIII

(#6c) Find the divergence and curl of $\mathbf{F} = \langle x^2yz,\ x^2z^2,\ 2x^2yz \rangle$. Then determine whether $\mathbf{F}$ is conservative and (if so) find a potential function U.

- Note $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$ and $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$.
- So $\nabla \cdot \mathbf{F} = \boxed{2xyz + 2x^2y}$, and $\nabla \times \mathbf{F} = \boxed{\langle 0, x^2y - 4xyz, 2xz^2 - x^2z \rangle}$.
- Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, field is $\boxed{\text{not conservative}}$.

Summary

We did some review problems for midterm 2.

Next lecture: Review for Midterm 3 (part 2)