Math 2321 (Multivariable Calculus)

Lecture #31 of 37 $\sim$ April 5, 2021

Green's Theorem

- Green's Theorem
- Tangential Form of Green's Theorem for Circulation
- Normal Form of Green's Theorem for Flux
- Applications of Green's Theorem

This material represents §4.5 from the course notes.

This is the last new material for Midterm 3; Wednesday and Thursday will be review.

Divergence and Curl Reminders

Recall divergence and curl, which I introduced last time:

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ *then the* *divergence* *of* $\mathbf{F}$ *is defined to be the scalar function* $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$.

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ *then the* *curl* *of* $\mathbf{F}$ *is defined to be the vector field*

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} =$$

$$\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle.$$

We can also define divergence and curl for vector fields in the plane: we simply pretend they have a z-coordinate that is zero.

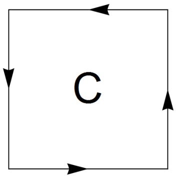
Green's Theorem, I

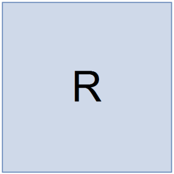
Green's Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus that relates a line integral of a function around a closed curve C to the double integral of a related function over the region R that is enclosed by the curve C.

Theorem (Green's Theorem)

If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then for any differentiable functions $P(x,y)$ and $Q(x,y)$, $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dy\,dx.$

Here is an example of a typical curve C and the region R it encloses:





Green's Theorem, II

The hypotheses about the curve ("simple closed rectifiable, oriented counterclockwise") are to ensure the curve is nice enough for the theorem to hold.

- "Simple" means that the curve does not cross itself.
- "Closed" means that its starting point is the same as its ending point (e.g., a circle).
- "Rectifiable" means it is piecewise-differentiable (i.e., differentiable except at a finite number of points).
- "Oriented counterclockwise" means that C runs around the boundary of R in the counterclockwise direction.

Green's Theorem, III

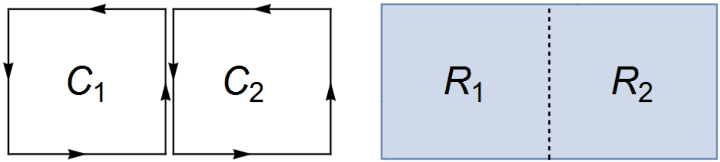
Proof (for rectangular regions):

- For a rectangular region $a \le x \le b$, $c \le y \le d$, we have $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$, where C_1, C_2, C_3, and C_4 are the four sides of the rectangle (with the proper orientation), and the function to be integrated on each curve is $P\,dx + Q\,dy$.
- Setting up parametrizations for the line integrals shows that $\int_{C_1} [P\,dx + Q\,dy] + \int_{C_3} [P\,dx + Q\,dy] = \int_a^b [P(x,c) - P(x,d)]\,dx$, $\int_{C_2} [P\,dx + Q\,dy] + \int_{C_4} [P\,dx + Q\,dy] = \int_c^d [Q(b,y) - Q(a,y)]\,dy$.
- We can also break the double integral into two parts: $\iint_R -\frac{\partial P}{\partial y}\,dy\,dx = \int_a^b \int_c^d -\frac{\partial P}{\partial y}\,dy\,dx = \int_a^b [P(x,c) - P(x,d)]\,dx$, $\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_d^c \int_a^b \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(b,y) - Q(a,y)]\,dy$.
- By comparing the expressions, we see that $$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx,$$ as desired.

Green's Theorem, III

To get Green's theorem for non-rectangular regions, we can break the region into a bunch of small rectangles, and then observe that Green's theorem is consistent with "gluing":

- If we have touching regions R_1 and R_2 with boundaries C_1 and C_2, then the union $R_1 \cup R_2$ has boundary $C_1 \cup C_2$, because the portions of C_1 and C_2 that align will have opposite directions and thus will cancel:



The line integral and the double integral are both continuous, so we may take limits of rectangular regions to get arbitrary ones.

Green's Theorem, IV

Green's Theorem can be used to convert line integrals on closed curves into double integrals.

- The double integrals are often much easier to evaluate if the curve is complicated but the region it encloses is simpler to describe.
- For example, if the boundary curve is a rectangle, then evaluating the line integral requires setting up four separate calculations, one for each side. But the double integral will be a single (easy!) calculation.

Green's Theorem, V

Example: If C is the counterclockwise boundary of the rectangle with $0 \leq x \leq 1$ and $0 \leq y \leq 2$, evaluate $\oint_C 3xy\,dx + x\,dy$.

- To compute the line integral as written, we need to parametrize each piece of the boundary. There are four pieces:

Green's Theorem, V

Example: If C is the counterclockwise boundary of the rectangle with $0 \le x \le 1$ and $0 \le y \le 2$, evaluate $\oint_C 3xy\,dx + x\,dy$.

- To compute the line integral as written, we need to parametrize each piece of the boundary. There are four pieces:
  1. The segment from $(0,0)$ to $(1,0)$.
  2. The segment from $(1,0)$ to $(1,2)$.
  3. The segment from $(1,2)$ to $(0,2)$.
  4. The segment from $(0,2)$ to $(0,0)$.
- Note that we need to preserve these orientations of the boundary segments, because the segments must be traversed in the counterclockwise direction around the boundary.

Green's Theorem, VI: Don't You Love Line Integrals?

Example: If C is the counterclockwise boundary of the rectangle with $0 \le x \le 1$ and $0 \le y \le 2$, evaluate $\oint_C 3xy\,dx + x\,dy$.

1. The segment from $(0,0)$ to $(1,0)$, parametrized by $x = t$, $y = 0$ for $0 \le t \le 1$. Then $dx = dt$ and $dy = 0\,dt$, so the integral here is $\int_0^1 0\,dt = 0$.
2. The segment from $(1,0)$ to $(1,2)$, parametrized by $x = 1$, $y = 2t$ for $0 \le t \le 1$. Then $dx = 0\,dt$ and $dy = 2\,dt$, so the integral here is $\int_0^1 2\,dt = 2$.
3. The segment from $(1,2)$ to $(0,2)$, parametrized by $x = 1 - t$, $y = 2$ for $0 \le t \le 1$. Then $dx = -dt$ and $dy = 0\,dt$, so the integral here is $\int_0^1 -6(1-t)\,dt = -3$.
4. The segment from $(0,2)$ to $(0,0)$, parametrized by $x = 0$, $y = 2 - 2t$ for $0 \le t \le 1$. Then $dx = 0$ and $dy = -2\,dt$, so the integral here is $\int_0^1 0\,dt = 0$.

The line integral is the sum of these four, which is -1.

Green's Theorem, VII

<u>Example</u>: If C is the counterclockwise boundary of the rectangle with $0 \leq x \leq 1$ and $0 \leq y \leq 2$, evaluate $\oint_C 3xy\,dx + x\,dy$.

- Let's now do this integral using Green's theorem.
- By Green's theorem,
$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy\,dx.$$
- Here, our region is the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 2$, and also $P = 3xy$ and $Q = x$.

Green's Theorem, VII

Example: If C is the counterclockwise boundary of the rectangle with $0 \le x \le 1$ and $0 \le y \le 2$, evaluate $\oint_C 3xy\,dx + x\,dy$.

- Let's now do this integral using Green's theorem.
- By Green's theorem,
$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx.$$
- Here, our region is the rectangle $0 \le x \le 1$ and $0 \le y \le 2$, and also $P = 3xy$ and $Q = x$.
- So since $Q_x - P_y = 1 - 3x$, the integral is
$$\int_0^1 \int_0^2 (1-3x)\,dy\,dx = \int_0^1 (2-6x)\,dx = (2x-3x^2)\Big|_{x=0}^{1} = -1.$$
- Note how much easier the calculation was using Green's theorem!

Green's Theorem, VIII

Example: Evaluate the integral $\oint_C 3x^2\,dx + 2xy\,dy$, where C is the counterclockwise boundary of the triangle having vertices $(0,0)$, $(1,0)$, and $(1,2)$.

Green's Theorem, VIII

<u>Example</u>: Evaluate the integral $\oint_C 3x^2\,dx + 2xy\,dy$, where C is the counterclockwise boundary of the triangle having vertices $(0,0)$, $(1,0)$, and $(1,2)$.

- We can use Green's theorem to do this as a double integral, rather than having to set up three separate line integrals.
- Green's Theorem says $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dy\,dx$, so setting $P = 3x^2$ and $Q = 2xy$ produces $\iint_R 2y\,dy\,dx$, where R is the interior of the triangle. A quick sketch shows that R is given by $0 \le x \le 1$ and $0 \le y \le 2x$.
- Thus, the double integral is $$\int_0^1 \int_0^{2x} 2y\,dy\,dx = \int_0^1 y^2\Big|_{y=0}^{2x} dx = \int_0^1 (2x)^2\,dx = \frac{4}{3}.$$

Green's Theorem, IX

<u>Example</u>: Evaluate the integral $\oint_C 3x^2\,dx + 2xy\,dy$, where C is the counterclockwise boundary of the triangle having vertices $(0,0)$, $(1,0)$, and $(1,2)$.

- In case you want to see the line integrals:

Green's Theorem, IX

<u>Example</u>: Evaluate the integral $\oint_C 3x^2\,dx + 2xy\,dy$, where C is the counterclockwise boundary of the triangle having vertices $(0,0)$, $(1,0)$, and $(1,2)$.

- In case you want to see the line integrals:
  1. The segment from $(0,0)$ to $(1,0)$, parametrized by $x = t$, $y = 0$ for $0 \le t \le 1$. Then $dx = dt$ and $dy = 0$, so the integral here is $\int_0^1 3t^2\,dt = 1$.
  2. The segment from $(1,0)$ to $(1,2)$, parametrized by $x = 1$, $y = t$ for $0 \le t \le 2$. Then $dx = 0$ and $dy = dt$, so the integral here is $\int_0^2 2t\,dt = 4$.
  3. The segment from $(1,2)$ to $(0,0)$, parametrized by $x = 1 - t$, $y = 2 - 2t$ for $0 \le t \le 1$. Then $dx = -dt$ and $dy = -2dt$, so the integral here is $\int_0^1 \left[3(1-t)^2(-dt) + 2(1-t)(2-2t)(-2dt)\right] = -11/3$.
- So the line integral is the sum $1 + 4 - 11/3 = 4/3$.

Green's Theorem, X

Example: Evaluate the integral $\oint_C -y^3\,dx + x^3\,dy$, where $C = C_1 \cup C_2 \cup C_3$ and C_1 is a line segment from $(0,0)$ to $(\sqrt{3},1)$, C_2 is the shorter circular arc along $x^2+y^2=4$ from $(\sqrt{3},1)$ to $(-2,0)$, and C_3 is a line segment from $(-2,0)$ to $(0,0)$.

Green's Theorem, X

Example: Evaluate the integral $\oint_C -y^3\,dx + x^3\,dy$, where $C = C_1 \cup C_2 \cup C_3$ and C_1 is a line segment from $(0,0)$ to $(\sqrt{3},1)$, C_2 is the shorter circular arc along $x^2+y^2=4$ from $(\sqrt{3},1)$ to $(-2,0)$, and C_3 is a line segment from $(-2,0)$ to $(0,0)$.

- We use Green's theorem here.
- Sketching the region shows that it is the interior of a circular sector with $0 \le r \le 2$ and $\pi/6 \le \theta \le \pi$, so we will set up the double integral in polar coordinates.
- Since $P = -y^3$ and $Q = x^3$, we have $Q_x - P_y = 3x^2 + 3y^2$.
- Thus, our integral is
$$\iint_R (3x^2+3y^2)\,dA = \int_{\pi/6}^{\pi}\int_0^2 3r^2\cdot r\,dr\,d\theta = \int_{\pi/6}^{\pi} 12\,d\theta = 10\pi.$$

Tangential and Normal Forms of Green's Theorem, I

We can use Green's Theorem to simplify the calculation of circulation and flux integrals on closed curves.

- Specifically, we can use the theorem to give expressions for circulation and flux either as line integrals or as double integrals over a region.
- Depending on the shape of the region and its boundary, and the nature of the field $\mathbf{F}$, either the line integral or the double integral can be easier.

Tangential and Normal Forms of Green's Theorem, I

First, for circulation:

Theorem (Green's Theorem, Tangential Form)

If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then the (counterclockwise) circulation around C is equal to $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$.

- This follows from our original statement of Green's theorem just by writing everything out: if $\mathbf{F} = \langle P, Q \rangle$ then $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$.
- So the tangential form of Green's Theorem reads $\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy\, dx$, which is exactly the statement that we gave earlier.

Tangential and Normal Forms of Green's Theorem, II

And now for flux:

Theorem (Green's Theorem, Normal Form)

If C is a simple closed rectifiable curve oriented counterclockwise, and R is the region it encloses, then the (outward normal) flux across C is equal to $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R (\operatorname{div} \mathbf{F})\, dA$.

- This also follows from our original statement of Green's theorem: if $\mathbf{F} = \langle P, Q \rangle$ then $\operatorname{div} \mathbf{F} = P_x + Q_y$.
- So the normal form of Green's Theorem reads $\oint_C -Q\, dx + P\, dy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dy\, dx$, which is the original statement of Green's theorem except with $-Q$ in place of P and P in place of Q.

Tangential and Normal Forms of Green's Theorem, III

Here's a nice interpretation of the normal form of Green's Theorem:

- Imagine that the vector field $\mathbf{F}$ is modeling population movement, and that C is the border of a country taking up the region R (where the only way in or out is via the border).
- At a city along the border C, the value $\mathbf{F} \cdot \mathbf{N}$ measures the immigration (in or out) to that city from across the border.
- At a city inside the country, the value $\operatorname{div} \mathbf{F}$ measures the net immigration (into or out of) that city.
- The normal form of Green's Theorem then says: if we add up the net immigration along the border, this equals the total population flow inside the country.
- These two quantities are definitely equal, since they both tally the net immigration into the country as a whole.

Tangential and Normal Forms of Green's Theorem, IV

Example: Let $\mathbf{F}(x,y) = \left\langle x^2 - 2xy,\, y^3 - x \right\rangle$, and let C be the counterclockwise boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$.

1. Find the circulation of $\mathbf{F}$ around C.
2. Find the outward normal flux of $\mathbf{F}$ across C.

Tangential and Normal Forms of Green's Theorem, IV

Example: Let $\mathbf{F}(x, y) = \langle x^2 - 2xy,\ y^3 - x \rangle$, and let C be the counterclockwise boundary of the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.

1. Find the circulation of $\mathbf{F}$ around C.
2. Find the outward normal flux of $\mathbf{F}$ across C.

- We could parametrize the boundary of this region and evaluate the line integrals to find the flux and circulation.
- However, this would be very tedious, as it requires computing four line integrals each time (one for each side of the square).
- We can save a lot of effort by using Green's Theorem, which applies because the boundary is a closed curve.
- Here, we have $P = x^2 - 2xy$ and $Q = y^3 - x$, and the region is $0 \le x \le 2$ and $0 \le y \le 2$.

Tangential and Normal Forms of Green's Theorem, V

<u>Example</u>: Let $\mathbf{F}(x, y) = \langle x^2 - 2xy,\, y^3 - x \rangle$, and let C be the counterclockwise boundary of the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.

1. Find the circulation of $\mathbf{F}$ around C.

Tangential and Normal Forms of Green's Theorem, V

Example: Let $\mathbf{F}(x, y) = \left\langle x^2 - 2xy,\ y^3 - x \right\rangle$, and let C be the counterclockwise boundary of the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.

1. Find the circulation of $\mathbf{F}$ around C.

- For circulation, the tangential form of Green's theorem says $\text{Circulation} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy\, dx.$
- Since $\frac{\partial Q}{\partial x} = -1$ and $\frac{\partial P}{\partial y} = -2x$, the circulation is $\int_0^2 \int_0^2 (-1 + 2x)\ dy\, dx = \int_0^2 (-2 + 4x)\ dx = 4.$

Tangential and Normal Forms of Green's Theorem, VI

Example: Let $\mathbf{F}(x, y) = \langle x^2 - 2xy,\ y^3 - x \rangle$, and let C be the counterclockwise boundary of the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.

2. Find the outward normal flux of $\mathbf{F}$ across C.

Tangential and Normal Forms of Green's Theorem, VI

<u>Example</u>: Let $\mathbf{F}(x, y) = \langle x^2 - 2xy,\ y^3 - x \rangle$, and let C be the counterclockwise boundary of the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.

2. Find the outward normal flux of $\mathbf{F}$ across C.

- For the flux, the normal form of Green's theorem says that
$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dy\, dx.$$
- Therefore, since $\dfrac{\partial P}{\partial x} = 2x - 2y$ and $\dfrac{\partial Q}{\partial y} = 3y^2$, the flux is
$$\int_0^2 \int_0^2 \left(2x - 2y + 3y^2\right)\, dy\, dx =$$
$$\int_0^2 \left(2xy - y^2 + y^3\right) \Big|_{y=0}^{2} dx = \int_0^2 (4x + 4)\, dx = 16.$$

Tangential and Normal Forms of Green's Theorem, VII

Example: Let $\mathbf{F}(x, y) = \langle 4x - x^2y,\, 2y + xy^2 \rangle$.

1. Find the circulation of $\mathbf{F}$ around the circle $x^2 + y^2 = 4$.
2. Find the outward normal flux of $\mathbf{F}$ across $x^2 + y^2 = 4$.

Tangential and Normal Forms of Green's Theorem, VII

Example: Let $\mathbf{F}(x, y) = \langle 4x - x^2y, 2y + xy^2 \rangle$.

1. Find the circulation of $\mathbf{F}$ around the circle $x^2 + y^2 = 4$.
2. Find the outward normal flux of $\mathbf{F}$ across $x^2 + y^2 = 4$.

- We can again use the normal and tangential forms of Green's Theorem: in this case, the region R is the interior of the circle $x^2 + y^2 \leq 4$.
- The integrals will be easiest to calculate if we set up in polar coordinates, since in polar the circle has the nice description $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Tangential and Normal Forms of Green's Theorem, VIII

Example: Let $\mathbf{F}(x, y) = \left\langle 4x - x^2 y,\, 2y + xy^2 \right\rangle$.

1. Find the circulation of $\mathbf{F}$ around the circle $x^2 + y^2 = 4$.

Tangential and Normal Forms of Green's Theorem, VIII

Example: Let $\mathbf{F}(x, y) = \langle 4x - x^2y,\, 2y + xy^2 \rangle$.

1. Find the circulation of $\mathbf{F}$ around the circle $x^2 + y^2 = 4$.

- By the tangential form of Green's theorem, the circulation is $\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$.
- The function is $Q_x - P_y = y^2 + x^2 = r^2$.
- Therefore, in polar coordinates, the integral is $\int_0^{2\pi} \int_0^2 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \Big|_{r=0}^{2} d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$.

Tangential and Normal Forms of Green's Theorem, IX

Example: Let $\mathbf{F}(x, y) = \langle 4x - x^2 y,\, 2y + xy^2 \rangle$.

2. Find the outward normal flux of $\mathbf{F}$ across $x^2 + y^2 = 4$.

Tangential and Normal Forms of Green's Theorem, IX

Example: Let $\mathbf{F}(x, y) = \langle 4x - x^2 y,\, 2y + xy^2 \rangle$.

2. Find the outward normal flux of $\mathbf{F}$ across $x^2 + y^2 = 4$.

- By the normal form of Green's theorem, the circulation is $\oint_C -Q\,dx + P\,dy = \iint_R (P_x + Q_y)\,dA$.
- The function is $P_x + Q_y = 4 + 2 = 6$.
- Therefore, in polar coordinates, the integral is $\int_0^{2\pi} \int_0^2 6 \cdot r\,dr\,d\theta = \int_0^{2\pi} 3r^2 \Big|_{r=0}^{2} d\theta = \int_0^{2\pi} 12\,d\theta = 24\pi$.

Applications of Green's Theorem, I

One application of Green's Theorem is to give ways to compute the area of a planar region using a line integral around its boundary.

- If C is the counterclockwise boundary curve of the region R and C and R satisfy the hypotheses of Green's Theorem, then

$$\text{Area of } R = \oint_C x\,dy = \oint_C -y\,dx = \oint_C \frac{1}{2}(x\,dy - y\,dx)$$

  because by Green's Theorem, each of the line integrals is equal to $\iint_R 1\,dy\,dx$, which is the area of R.
- You may have seen these formulas in single-variable calculus: the area enclosed by $x = x(t)$, $y = y(t)$ is given by $\int_a^b y(t) \cdot x'(t)\,dt = \int_C y\,dx$, or $\int_a^b -x(t) \cdot y'(t)\,dt = \int_C -x\,dy$.
- Also, the area inside a polar graph $r = r(\theta)$ is $\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2\,d\theta$: this follows by writing the third formula in polar coordinates.

Applications of Green's Theorem, II

Example: Compute the area enclosed by the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

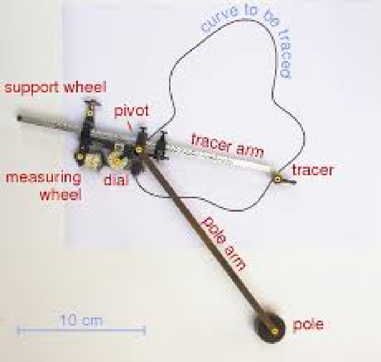
- Here we can use the formula Area of $R = \oint_C \frac{1}{2}(x\,dy - y\,dx)$.
- So then we see $A = \displaystyle\oint_C \frac{1}{2}(x\,dy - y\,dx)$

  $= \displaystyle\int_0^{2\pi} \frac{1}{2}\left[(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\right] dt$

  $= \displaystyle\int_0^{2\pi} \frac{ab}{2}\,dt = \pi ab.$

Applications of Green's Theorem, III

One physical application of this idea is the construction of planimeters: they are devices used for measuring the area of a region that operate by tracing along its boundary.



(Courtesy: American Mathematical Society)

The basic principle is that the planimeter measures the amount of movement perpendicular to its measuring arm: integrating the resulting dot product around the boundary of the curve, per Green's theorem, then yields the area.

Applications of Green's Theorem, IV

Another application of Green's theorem is to establish one of our characterizations of conservative vector fields in the plane.

- As noted last class, $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if the circulation around any closed curve is zero.
- By Green's theorem, this is equivalent to saying that $\iint_R \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k}\, dA = 0$ for every region R.
- But, because the function $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} = Q_x - P_y$ is continuous, the only way it could integrate to zero on every possible region is if it is actually zero everywhere.
- Thus, $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if its curl is zero, is claimed.

Summary

We discussed Green's theorem and its applications.

We described the normal and tangential forms of Green's theorem, and how to use them to find circulation and flux.

Next lecture: Review for midterm 3.