Math 2321 (Multivariable Calculus)

Lecture #30 of 37 $\sim$ April 1st, 2021

Path-Independence, Conservative Fields, Potential Functions

- Path-Independence of Work Integrals
- The Fundamental Theorem of Line Integrals
- Conservative Vector Fields and Potential Functions
- Divergence and Curl

This material represents §4.4 from the course notes.

Work Reminder

Today we will discuss more with work integrals in 2 and 3 dimensions.

- Remember that the work done by a vector field $\mathbf{F}$ on a particle that travels along a plane curve C is $\text{Work} = \int_C P\,dx + Q\,dy$, or along a space curve it is $\text{Work} = \int_C P\,dx + Q\,dy + R\,dz$.

What we will investigate now is whether there is any relation between the work integrals $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for two different curves C_1 and C_2.

Path Independence, I

<u>Example</u>: Let $\mathbf{F}(x, y) = \langle y, x \rangle$. Evaluate the work integrals from $(0, 0)$ to $(1, 1)$ along the paths

1. $C_1 : (x, y) = (t, t)$, for $0 \leq t \leq 1$.
2. $C_2 : (x, y) = (t^3, t^2)$, for $0 \leq t \leq 1$.
3. $C_3 : (x, y) = (t^7,\ t^{10})$, for $0 \leq t \leq 1$.

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- On C_1, $\mathbf{F} = \langle t, t \rangle$, so the work is $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [t \cdot 1 + t \cdot 1]\, dt = \int_0^1 2t\, dt = 1.$
- On C_2, $\mathbf{F} = \left\langle t^2, t^3 \right\rangle$, so the work is $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[t^2 \cdot 3t^2 + t^3 \cdot 2t\right] dt = \int_0^1 5t^4\, dt = 1.$
- On C_3, $\mathbf{F} = \left\langle t^{10}, t^7 \right\rangle$, so the work is $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[t^{10} \cdot 7t^6 + t^7 \cdot 10t^9\right] dt = \int_0^1 17t^{16}\, dt = 1.$
- Notice all three yield the same value!

Path Independence, II

Example: Let $\mathbf{G}(x, y) = \langle y^2, x \rangle$. Evaluate the work integrals from $(0, 0)$ to $(1, 1)$ along the paths

1. $C_1 : (x, y) = (t, t)$, for $0 \le t \le 1$.
2. $C_2 : (x, y) = (t^3, t^2)$, for $0 \le t \le 1$.
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Path Independence, II

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- On C_1, $\mathbf{G} = \langle t^2, t \rangle$, so the work is $\int_{C_1} \mathbf{G} \cdot d\mathbf{r} = \int_0^1 \left[t^2 \cdot 1 + t \cdot 1 \right] dt = \int_0^1 (t^2 + t)\, dt = 5/6$.
- On C_2, $\mathbf{G} = \langle t^4, t^3 \rangle$, so the work is $\int_{C_2} \mathbf{G} \cdot d\mathbf{r} = \int_0^1 \left[t^4 \cdot 3t^2 + t^3 \cdot 2t \right] dt = \int_0^1 (3t^6 + 2t^4)\, dt = 29/35$.
- On C_3, $\mathbf{G} = \langle t^{20}, t^7 \rangle$, so the work is $\int_{C_3} \mathbf{G} \cdot d\mathbf{r} = \int_0^1 \left[t^{30} \cdot 7t^6 + t^7 \cdot 10t^9 \right] dt = \int_0^1 (7t^{36} + 10t^{16})\, dt = 389/459$.
- Notice all three yield different values!

Path Independence, III

So, our question is: why is it that all of the work integrals for $\mathbf{F}(x, y) = \langle y, x \rangle$ had the same value, but the work integrals for $\mathbf{G}(x, y) = \langle y^2, x \rangle$ didn't?

- It might be that I just chose misleading examples, and in fact for $\mathbf{F}$ there are some other paths from $(0, 0)$ to $(1, 1)$ where the value of the work integral is different.
- This turns out not to be the case (if you like, you can try some other paths!).
- But it's not so obvious why that would be, nor what causes the difference in the behavior between $\mathbf{F}$ and $\mathbf{G}$.

Conservative Fields, I

Let's define the property we're interested in, and then try to characterize it in other ways:

Definition

A vector field $\mathbf{F}$ *is conservative on a region* R *if, for any two paths* C_1 *and* C_2 *(inside* R*) from* P_1 *to* P_2*, it is true that* $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$*. In other words,* $\mathbf{F}$ *is conservative if any two paths with the same endpoints yield the same work integral.*

Conservative Fields, II

Here's a starting point: $\mathbf{F}$ is conservative on a region R if, for any closed curve C in R, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (A closed curve is one whose start and end points are the same.)

- Notation: For a line integral around a closed curve, we often use the notation $\oint_C$, the circle being a suggestive example of a closed curve.

Conservative Fields, II

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- Notation: For a line integral around a closed curve, we often use the notation $\oint_C$, the circle being a suggestive example of a closed curve.
- The statement above is equivalent to the definition from the previous slide because, if C_1 and C_2 are two paths from P_1 to P_2, then we can construct a closed path C by following C_1 from P_1 to P_2 and then following C_2 from P_2 back to P_1.
- Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, and so the left-hand side is zero if and only if the right-hand side is zero.

Conservative Fields, III

It turns out that we can give a simple but very useful criterion for when a vector field is conservative:

Theorem (Fundamental Theorem of Calculus for Line Integrals)

The vector field $\mathbf{F}$ *is conservative on a simply-connected region* R *if and only if there exists a function* U*, called a* <u>*potential function*</u> *for* $\mathbf{F}$*, such that* $\mathbf{F} = \nabla U$*. If such a function* U *exists, then* $\int_a^b \mathbf{F} \cdot d\mathbf{r} = U(b) - U(a)$ *along any path from* a *to* b*.*

Notice the similarity of the statement $\int_a^b \mathbf{F} \cdot d\mathbf{r} = U(b) - U(a)$ to the Fundamental Theorem of Calculus, which relates the integral of a derivative of a function to its values at the endpoints of a path.

Conservative Fields, IV

A technical note about the hypotheses of the theorem:

- The term "simply-connected" is a technical requirement needed for the proof of the theorem: intuitively, a simply-connected region consists of a single piece that does not have any "holes" in it.
- More rigorously, it means that the region is connected (contains only one "piece") and that if we take any closed loop in the region, we can continuously shrink it to a point without leaving the region.
- The disc $x^2 + y^2 \leq 4$ is simply-connected, whereas the annulus $1 \leq x^2 + y^2 \leq 4$ is not.

We will show one direction of the proof. (The other we will essentially deduce during the next lecture.)

Conservative Fields, V

Proof (Reverse Direction in 3-Space):

- Suppose $\mathbf{F} = \nabla U$.
- By the multivariable chain rule, if C is the path with $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \leq t \leq b$, then
$$\frac{dU}{dt} = \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}.$$
- Then, by the Fundamental Theorem of Calculus, we see
$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\ &= \int_a^b \left[\frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial U}{\partial z} \cdot \frac{dz}{dt} \right] dt \\ &= \int_a^b \left[\frac{dU}{dt} \right] dt = U(\mathbf{r}(b)) - U(\mathbf{r}(a)) \end{aligned}$$

  and so $\mathbf{F}$ is conservative.

Conservative Fields, VI

The reason for the name "potential function" is because U behaves like a potential-energy function when we interpret $\mathbf{F}$ as a vector field doing work on a particle.

- Specifically, if $\mathbf{F} = \nabla U$, then the work done by the field $\mathbf{F}$ on a particle traveling from a to b is equal to $U(b) - U(a)$.
- Another way of saying this is: the sum of [the work done by $\mathbf{F}$ in moving a particle from the origin to a point P] with [the value $-U(P)$] is the same for all points P.
- In this guise, the fundamental theorem of line integrals is a conservation of energy statement: the work represents kinetic energy, while the value $-U(P)$ represents the potential energy at P. The sum of these two energies is constant.

Conservative Fields, VII

If we can see that a vector field is conservative, then it is very easy to compute work integrals: we just need to find a potential function for the vector field.

- Sometimes, you can spot a potential function just by trial and error.
- In a little while, we will discuss a more systematic method for finding potential functions.

Conservative Fields, VIII

Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle 2x + y, x \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \left\langle -2\cos(\pi e^t), \tan^{-1}(t) \right\rangle$ from $t = 0$ to $t = 1$.

- If we try to set up the integral directly using the parametrization, it will be rather unpleasant.
- However, this vector field is conservative: it is not hard to see that for $U(x, y) = x^2 + xy$, we have $\nabla U = \langle 2x + y, x \rangle = \mathbf{F}$.

Conservative Fields, VIII

Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle 2x + y, x \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \left\langle -2\cos(\pi e^t), \tan^{-1}(t) \right\rangle$ from $t = 0$ to $t = 1$.

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- However, this vector field is conservative: it is not hard to see that for $U(x, y) = x^2 + xy$, we have $\nabla U = \langle 2x + y, x \rangle = \mathbf{F}$.
- By the Fundamental Theorem of Calculus for line integrals, the work done by the vector field is then simply the value of $U(\mathbf{r}(1)) - U(\mathbf{r}(0))$.
- Since $\mathbf{r}(1) = \langle -2\cos(\pi e), \pi/4 \rangle$ and $\mathbf{r}(0) = \langle -2, 0 \rangle$, the work is $U(2, \pi/4) - U(-2, 0) = 4\cos^2(\pi e) - 2\cos(\pi e) \cdot \pi/4 - 4$.

Conservative Fields, IX

Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle 3x^2, 2y \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \langle 11t^7 - 10t^{19}, \sin^8(\pi\sqrt{t}) \rangle$ from $t = 0$ to $t = 1$.

Conservative Fields, IX

Example: Find the work done by the vector field $\mathbf{F}(x,y) = \langle 3x^2, 2y \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \langle 11t^7 - 10t^{19}, \sin^8(\pi\sqrt{t}) \rangle$ from $t = 0$ to $t = 1$.

- This vector field is conservative: for $U(x,y) = x^3 + y^2$, we have $\nabla U = \langle 3x^2, 2y \rangle = \mathbf{F}$.
- By the Fundamental Theorem of Calculus for line integrals, the work done by the vector field is then simply the value of $U(\mathbf{r}(1)) - U(\mathbf{r}(0))$.
- Since $\mathbf{r}(1) = \langle 1, 0 \rangle$ and $\mathbf{r}(0) = \langle 0, 0 \rangle$, the work is $U(1,0) - U(0,0) = 3$.

Divergence and Curl, I

We would like to be able to determine easily whether a given vector field is conservative. To do this, we require a preliminary definition of a quantity known as the curl of a vector field.

- Since it has a similar definition and we will be using it later, we may as well also define the divergence now as well.
- Both of these quantities are defined in terms of the differential operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.
- Indeed, we have already used this operator in the past to define the gradient of a function. There, we have $\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle f_x, f_y, f_z \rangle$.

Divergence and Curl, II

The divergence of a vector field is a scalar function:

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ *then the* *divergence* *of* $\mathbf{F}$ *is defined to be the scalar function* $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$.

The curl of a vector field is a vector field:

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ *then the* *curl* *of* $\mathbf{F}$ *is defined to be the vector field*

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} =$$

$$\left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle.$$

Divergence and Curl, III

Example: Let $\mathbf{F} = \langle 3x^2y, xyz, e^{xy} \rangle$.

1. Calculate the divergence of $\mathbf{F}$.
2. Calculate the curl of $\mathbf{F}$.

Divergence and Curl, III

Example: Let $\mathbf{F} = \left\langle 3x^2y, xyz, e^{xy} \right\rangle$.

1. Calculate the divergence of $\mathbf{F}$.
2. Calculate the curl of $\mathbf{F}$.

- The divergence of $\langle P, Q, R \rangle$ is $P_x + Q_y + R_z$.
- So $\mathrm{div}(\mathbf{F}) = 6xy + xz + 0$.
- The curl of $\langle P, Q, R \rangle$ is $\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$. The best way to remember this, by the way, is to write down the determinant mnemonic.
- So $\mathrm{curl}\ \mathbf{F} = \nabla \times \mathbf{F} = \left\langle xe^{xy} - xy,\ 0 - ye^{xy},\ yz - 3x^2 \right\rangle$.

Divergence and Curl, IV

We can also define divergence and curl for vector fields in the plane. We simply pretend they have a z-coordinate that is zero.

Example: Let $\mathbf{F} = \langle x^3 y^2, \sin(xy) \rangle$.

1. Calculate the divergence of $\mathbf{F}$.
2. Calculate the curl of $\mathbf{F}$.

Divergence and Curl, IV

We can also define divergence and curl for vector fields in the plane. We simply pretend they have a z-coordinate that is zero.

Example: Let $\mathbf{F} = \langle x^3y^2, \sin(xy) \rangle$.

1. Calculate the divergence of $\mathbf{F}$.
2. Calculate the curl of $\mathbf{F}$.

- The divergence of $\langle P, Q, 0\rangle$ is $P_x + Q_y$.
- So $\mathrm{div}(\mathbf{F}) = 3x^2y^2 + x\cos(xy)$.
- The curl of $\langle P, Q, 0\rangle$ is $\langle 0, 0, Q_x - P_y\rangle$, when P and Q do not depend on z.
- So $\mathrm{curl}\ \mathbf{F} = \nabla \times \mathbf{F} = \langle 0, 0, y\cos(xy) - 2x^3y\rangle$.

Divergence and Curl, V

It is quite reasonable at this point to wonder why I am giving you these definitions of divergence and the curl now.

- The reason is that they will be very important in the next few lectures, so I wanted you to be familiar with them now.
- It is also quite reasonable to wonder what exactly the divergence and the curl represent and why they are important. Unfortunately, I cannot tell you the answer without spoiling the main results of the rest of the lectures, so you'll just have to wait a few classes to find out!
- But they have very concrete interpretations in terms of fluid flow.

Conservative Fields and Curl, I

The main result is that the curl of a vector field determines whether or not it is conservative:

Theorem (Zero Curl Implies Conservative)

A vector field on a simply-connected region in the plane or in 3-space is conservative if and only if its curl is zero. More explicitly, we have the following:

1. *The vector field* $\mathbf{F} = \langle P, Q \rangle$ *is conservative on a simply-connected region* R *in the plane if and only if* $P_y = Q_x$*.*
2. *The vector field* $\mathbf{F} = \langle P, Q, R \rangle$ *is conservative on a simply-connected region* D *in 3-space if and only if* $P_y = Q_x$*,* $P_z = R_x$*, and* $Q_z = R_y$*.*

Conservative Fields and Curl, II

It is fairly easy to see why a conservative field requires the equality of the derivatives of the components.

- Explicitly, if $\mathbf{F} = \langle P, Q \rangle = \nabla U$ then $P = U_x$ and $Q = U_y$, so by the equality of mixed partial derivatives, we see that $P_y = U_{xy} = U_{yx} = Q_x$.
- The three necessary equalities when $\mathbf{F} = \langle P, Q, R \rangle$ follow in the same way: if $\mathbf{F} = \nabla U$ then $P = U_x$, $Q = U_y$, and $R = U_z$, so $P_y = U_{xy} = U_{yx} = Q_x$, $P_z = U_{xz} = U_{zx} = R_x$, and $Q_z = U_{yz} = U_{zy} = R_y$.
- The converse statement (that zero curl implies the field actually is conservative) is more difficult, and we will skip it for now – in fact, it follows from the results we will cover next.

Conservative Fields and Curl, II

Our theorems give us an effective procedure for determining whether a field is conservative: we first check whether its curl is zero, and then (if it is) we can try to find a potential function by computing antiderivatives.

- If the field has nonzero curl, we automatically know it is not conservative.
- If the field has zero curl, we know it is conservative, and that there exists a function U with $\mathbf{F} = \nabla U$.
- We can then try to identify U by taking antiderivatives of the components of $\mathbf{F}$.
- The only tricky part is that we may have to piece together the shape of U from all of the partial derivatives, in case there are terms that don't involve all of the variables.

Conservative Fields and Curl, III

Example: Determine whether $\mathbf{F}(x,y) = \langle x^2 + y, x + y^2 \rangle$ is conservative, and if so, find a potential function.

Conservative Fields and Curl, III

<u>Example</u>: Determine whether $\mathbf{F}(x, y) = \left\langle x^2 + y, x + y^2 \right\rangle$ is conservative, and if so, find a potential function.

- We see $\frac{\partial}{\partial y}\left[x^2 + y\right] = 1 = \frac{\partial}{\partial x}\left[x + y^2\right]$, so $\mathbf{F}$ is conservative.
- To find a potential function U with $\nabla U = \mathbf{F}$, we need to find U such that $U_x = x^2 + y$ and $U_y = x + y^2$.

Conservative Fields and Curl, III

<u>Example</u>: Determine whether $\mathbf{F}(x, y) = \left\langle x^2 + y, x + y^2 \right\rangle$ is conservative, and if so, find a potential function.

- We see $\frac{\partial}{\partial y}\left[x^2 + y\right] = 1 = \frac{\partial}{\partial x}\left[x + y^2\right]$, so $\mathbf{F}$ is conservative.
- To find a potential function U with $\nabla U = \mathbf{F}$, we need to find U such that $U_x = x^2 + y$ and $U_y = x + y^2$.
- Taking the antiderivative of $U_x = x^2 + y$ with respect to x yields $U = \frac{1}{3}x^3 + xy + f(y)$, for some function $f(y)$.
- To find $f(y)$ we differentiate: $U_y = x + f'(y)$, so we get $f'(y) = y^2$ so we can take $f(y) = \frac{1}{3}y^3$.
- So a potential function for $\mathbf{F}$ is $U(x, y) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3$.

Conservative Fields and Curl, IV

Example: Determine whether $\mathbf{G}(x, y) = \langle x + y^2, x^2 + y \rangle$ is conservative, and if so, find a potential function.

Conservative Fields and Curl, IV

Example: Determine whether $\mathbf{G}(x, y) = \langle x + y^2, x^2 + y \rangle$ is conservative, and if so, find a potential function.

- For $\mathbf{G}$, we see $\dfrac{\partial}{\partial y}\left[x + y^2\right] = 2y$, while $\dfrac{\partial}{\partial x}\left[x^2 + y\right] = 2x$.
- These are not equal, so the field is not conservative.

Conservative Fields and Curl, V

Example: Determine if $\mathbf{H}(x, y, z) = \langle y + 2z, x + 3z, 2x + 3y \rangle$ is conservative, and if so, find a potential function.

Conservative Fields and Curl, V

Example: Determine if $\mathbf{H}(x, y, z) = \langle y + 2z, x + 3z, 2x + 3y \rangle$ is conservative, and if so, find a potential function.

- For $\mathbf{H}$, we have $\frac{\partial}{\partial y}[y + 2z] = 1 = \frac{\partial}{\partial x}[x + 3z]$, $\frac{\partial}{\partial z}[y + 2z] = 2 = \frac{\partial}{\partial x}[2x + 3y]$, and $\frac{\partial}{\partial z}[x + 3z] = 3 = \frac{\partial}{\partial y}[2x + 3y]$, so the field is conservative.

Conservative Fields and Curl, VI

Example: Determine if $\mathbf{H}(x, y, z) = \langle y + 2z, x + 3z, 2x + 3y \rangle$ is conservative, and if so, find a potential function.

- To find a potential function U with $\nabla U = \mathbf{H}$, we need to find U such that $U_x = y + 2z$, $U_y = x + 3z$, and $U_z = 2x + 3y$.

Conservative Fields and Curl, VI

Example: Determine if $\mathbf{H}(x, y, z) = \langle y + 2z, x + 3z, 2x + 3y \rangle$ is conservative, and if so, find a potential function.

- To find a potential function U with $\nabla U = \mathbf{H}$, we need to find U such that $U_x = y + 2z$, $U_y = x + 3z$, and $U_z = 2x + 3y$.
- Taking the antiderivative of $U_x = y + 2z$ with respect to x yields $U = xy + 2xz + f(y, z)$, for some function $f(y, z)$.
- To find $f(y, z)$ we differentiate: $x + f_y = x + 3z$ and $2x + f_z = 2x + 3y$, so $f_y = 3z$ and $f_z = 3y$. Repeating the process yields $f = 3yz + g(z)$, where $g'(z) = 0$.
- Thus we see that a potential function for $\mathbf{H}$ is $U(x, y, z) = xy + 2xz + 3yz$.

Conservative Fields and Curl, VII

Example: Determine if $\mathbf{F}(x, y, z) = \left\langle 3x^2yz^2, x^3z^2 + 2y - z, 2x^3yz - y + 4z \right\rangle$ is conservative, and if so, find a potential function.

Conservative Fields and Curl, VII

Example: Determine if $\mathbf{F}(x, y, z) = \left\langle 3x^2yz^2, x^3z^2 + 2y - z, 2x^3yz - y + 4z \right\rangle$ is conservative, and if so, find a potential function.

- For $\mathbf{F}$, we have $\frac{\partial}{\partial y}\left[3x^2yz^2\right] = 3x^2z^2 = \frac{\partial}{\partial x}\left[x^3z^2 + 2y - z\right]$, $\frac{\partial}{\partial z}\left[3x^2yz^2\right] = 6x^2yzy = \frac{\partial}{\partial x}\left[2x^3yz - y + 4z\right]$, and $\frac{\partial}{\partial z}\left[x^3z^2 + 2y - z\right] = 2x^3z - 1 = \frac{\partial}{\partial y}\left[2x^3yz - y + 4z\right]$, so the field is conservative.

Conservative Fields and Curl, VIII

Example: Determine if $\mathbf{F}(x, y, z) = \langle 3x^2yz^2, x^3z^2 + 2y - z, 2x^3yz - y + 4z \rangle$ is conservative, and if so, find a potential function.

- To find a potential function U with $\nabla U = \mathbf{F}$, we need to find U such that $U_x = 3x^2yz^2$, $U_y = x^2z^2 + 2y - z$, and $U_z = 2x^3yz - y + 4z$.

Conservative Fields and Curl, VIII

Example: Determine if $\mathbf{F}(x,y,z) = \langle 3x^2yz^2, x^3z^2 + 2y - z, 2x^3yz - y + 4z \rangle$ is conservative, and if so, find a potential function.

- To find a potential function U with $\nabla U = \mathbf{F}$, we need to find U such that $U_x = 3x^2yz^2$, $U_y = x^2z^2 + 2y - z$, and $U_z = 2x^3yz - y + 4z$.
- Looking at U_x, we see that we need a term x^3yz^2 in U.
- If $U = x^3yz^2$ then this accounts for the x^2z^2 term in U_y, but it still needs a $2y - z$, which we can get by adding $y^2 - yz$ to U.
- If $U = x^3yz^2 + y^2 - yz$ then U_x and U_y are correct, but U_z is missing the $+4z$, which we can get by adding $2z^2$ to U.
- So, finally, we get $U = x^3yz^2 + y^2 - yz + 2z^2$, which works.

Conservative Fields and Curl, IX

Example: For the vector field $\mathbf{F}(x, y, z) = \left\langle 3x^2yz^2, x^3z^2 + 2y - z, 2x^3yz - y + 4z \right\rangle$, find the work done by $\mathbf{F}$ on a particle that travels along the curve $C : \mathbf{r}(t) = \left\langle \sin(\pi t),\ t\sqrt{t+3},\ 2t^3 + 2 \right\rangle$ for $0 \le t \le 1$.

Conservative Fields and Curl, IX

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- We can use the potential function $U = x^3yz^2 - y^2 + yz + 2z^2$ we just calculated.
- By the fundamental theorem of line integrals, the work is then $U(\mathbf{r}(1)) - U(\mathbf{r}(0)) = U(0, 2, 4) - U(0, 0, 2) = 36 - 0 = 36$.

Summary

We discussed path independence of work integrals, conservative vector fields, and potential functions.

We established the fundamental theorem for line integrals.

We introduced the divergence and curl of a vector field.

We discussed how to establish whether a vector field is conservative by computing its curl.

Next lecture: Green's theorem.