Math 2321 (Multivariable Calculus)

Lecture #29 of 37 ~ March 31, 2021

Flux Across Surfaces

- Circulation and Flux
- Flux Across Surfaces

This material represents §4.3.3 from the course notes.

Circulation and Flux Reminders, I

Last time, we discussed circulation and flux, which are two natural quantities that arise when studying vector fields representing fluid flow.

- To visualize these, imagine you are riding a bicycle on a windy day.
- The circulation measures how much the wind is helping or hindering you: in other words, how much it is pushing you tangentially along your path of motion.
- The flux measures how much the wind is blowing you off course: in other words, how much it is pushing normally across your path of motion.

Circulation and Flux Reminders, II

In the plane, we can compute circulation and flux as line integrals:

Definition

Suppose $\mathbf{F} = \langle P, Q \rangle$ *is a vector field representing the velocity of a fluid flowing through the plane.*
The <u>*circulation*</u> *(or* <u>*flow*</u>*) of the vector field* $\mathbf{F}$ *along the curve* C *is*

$$\text{Circulation} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C P\, dx + Q\, dy,$$

where $\mathbf{T}$ *is the unit tangent vector to the curve* C*.*
The <u>*flux*</u> *of the vector field* $\mathbf{F}$ *across the curve* C *is*

$$\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_C -Q\, dx + P\, dy,$$

where $\mathbf{N}$ *is the unit normal vector to the curve* C*.*

Flux Across Surfaces, I

In 3-space, the notion of circulation along a curve remains essentially the same as in the plane.

- If $\mathbf{F} = \langle P, Q, R \rangle$, the circulation of $\mathbf{F}$ along the curve C is $\int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left[P\,\frac{dx}{dt} + Q\,\frac{dy}{dt} + R\,\frac{dz}{dt} \right] dt.$

However, the physical interpretation of flux in 3-space means that we must measure fluid flow across a surface, rather than a curve.

- The resulting flux integral is then a surface integral, rather than a line integral.
- Instead of measuring how much $\mathbf{F}$ aligns with the unit normal vector $\mathbf{N}$ to the curve C, we want to measure how much $\mathbf{F}$ aligns with the unit normal vector $\mathbf{n}$ to a surface S.
- Thus, we want to integrate the dot product $\mathbf{F} \cdot \mathbf{n}$ on the surface S, where $\mathbf{n}$ represents the unit normal vector to S.

Flux Across Surfaces, II

Our analysis indicates that the total flux of a vector field $\mathbf{F}$ across a surface S is the integral of $\mathbf{F} \cdot \mathbf{n}$ over the surface.

Definition

If $\mathbf{F}$ represents the velocity of a fluid flowing through 3-space, then the (outward normal) flux of the vector field $\mathbf{F}$ across the surface S is given by the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

where $\mathbf{n}$ is the outward unit normal vector to the surface.
The flux measures the total amount of fluid flowing across S.

As with the circulation and flux integrals in the plane, we (usually) do not want to have to calculate the normal vector $\mathbf{n}$ explicitly.

Flux Across Surfaces, III

Some comments about notation and terminology:

- When speaking of a unit normal vector to a surface we will use a lowercase **n**, to keep the notation different from the unit normal **N** to a curve (which is an uppercase **N**).
- The unit normal vector to a surface is defined to be the normal vector of the tangent plane.

The integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ computes the flux through the surface in the direction of the outward normal vector to the surface.

- All of this is assuming that there is a coherent notion of an "outward normal vector". This may seem like a reasonable expectation, but some surfaces, like the Möbius strip, cannot be consistently assigned a normal vector.
- We will therefore assume, for our discussions, that all our surfaces are "orientable", meaning that there is a continuous assignment of a normal vector to all points on the surface.

Flux Across Surfaces, IV

Since **n** is the normal vector to the surface's tangent plane, we can write it down explicitly, and thus set up the surface integral.

- If S is parametrized by $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$, then a normal vector is given by the cross product $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$, so we get a unit normal vector $\mathbf{n} = \left(\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right) \Big/ \left|\left|\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}\right|\right|$.
- If S is an implicit surface $g(x,y,z) = c$, then a normal vector is given by the gradient ∇g, so we get a unit normal vector $\mathbf{n} = \nabla g / \left|\left|\nabla g\right|\right|$.
- By plugging these expressions into the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, we obtain explicit formulas for the outward normal flux across a surface S.

Flux Across Surfaces, V

First, for a parametric surface:

Proposition (Flux Across a Parametric Surface)

Suppose $\mathbf{F}$ *is a vector field and* S *is a surface parametrized by* $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$ *for* s *and* t *in a region* R*. Then the outward normal flux of* $\mathbf{F}$ *across* S *is equal to*

$$Flux = \iint_R \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right) ds\, dt$$

provided that $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$ *is the outward-pointing normal vector of* S*.*

Pleasantly, the denominator for the unit normal vector $\mathbf{n} = \left(\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} \right) \Big/ \left|\left| \dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} \right|\right|$ cancels the factor $\left|\left| \dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} \right|\right|$ that comes from the surface area differential.

Flux Across Surfaces, VI

Also, for an implicit surface:

Proposition (Flux Across an Implicit Surface)

Suppose $\mathbf{F}$ *is a vector field and* S *is a portion of the surface defined implicitly by* $g(x, y, z) = c$*, where* R *is the projection of* S *in the* xy*-plane. Then the outward normal flux of* $\mathbf{F}$ *across* S *is equal to*

$$Flux = \iint_R \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{k}|} \, dy \, dx$$

provided that $\partial g/\partial z$ *is nonzero on* R*. (Note here that the denominator term* $\nabla g \cdot \mathbf{k}$ *is simply the partial derivative* $\partial g/\partial z$*.)*

We also get a cancellation of the unpleasant denominator term $||\nabla g||$ in this formula.

Flux Across Surfaces, VII

Both of the formulas follow just by writing down the dot product $\mathbf{F} \cdot \mathbf{n}$ as a function and setting up the appropriate surface integral.

- Depending on the description of the surface, either of these two methods (i.e., via a parametrization or as an implicit surface) may be more convenient for computing a flux integral.

We also mention that, occasionally, the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ is written as $\iint_S \mathbf{F} \cdot d\boldsymbol{\sigma}$.

- Here, $\boldsymbol{\sigma}$ is being considered as a vector differential.
- The resulting flux integral is then called "the integral of the vector field $\mathbf{F}$ on the surface S".
- We will always refer to this integral explicitly as a flux integral, using our regular surface integral notation $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$.

Flux Across Surfaces, VIII

Example: Consider the vector field $\mathbf{F} = \left\langle xz^2, yz^2, x^3e^y \right\rangle$ on the portion of the cylinder $x^2 + y^2 = 4$ between $z = -1$ and $z = 1$.

1. Find a parametrization for this portion of the cylinder.
2. Find the outward normal vector to the cylinder.
3. Set up and evaluate the flux of $\mathbf{F}$ across S.

Flux Across Surfaces, VIII

Example: Consider the vector field $\mathbf{F} = \left\langle xz^2, yz^2, x^3 e^y \right\rangle$ on the portion of the cylinder $x^2 + y^2 = 4$ between $z = -1$ and $z = 1$.

1. Find a parametrization for this portion of the cylinder.
2. Find the outward normal vector to the cylinder.
3. Set up and evaluate the flux of $\mathbf{F}$ across S.

- From cylindrical coordinates, we can parametrize the cylinder as $\mathbf{r}(s, t) = \langle 2\cos t, 2\sin t, s\rangle$.
- The desired portion corresponds to $-1 \le s \le 1$ and $0 \le t \le 2\pi$.

Flux Across Surfaces, IX

Example: Consider the vector field $\mathbf{F} = \left\langle xz^2, yz^2, x^3 e^y \right\rangle$ on the portion of the cylinder $x^2 + y^2 = 4$ between $z = -1$ and $z = 1$.

2. Find the outward normal vector to the cylinder.

- Since $\mathbf{r}(s, t) = \langle 2\cos t, 2\sin t, s\rangle$,

Flux Across Surfaces, IX

Example: Consider the vector field $\mathbf{F} = \left\langle xz^2, yz^2, x^3 e^y \right\rangle$ on the portion of the cylinder $x^2 + y^2 = 4$ between $z = -1$ and $z = 1$.

2. Find the outward normal vector to the cylinder.

- Since $\mathbf{r}(s,t) = \langle 2\cos t, 2\sin t, s\rangle$, we see that $\partial\mathbf{r}/\partial t = \langle -2\sin t,\ 2\cos t,\ 0\rangle$ and $\partial\mathbf{r}/\partial s = \langle 0, 0, 1\rangle$.
- Thus, the normal vector is

$$\frac{\partial\mathbf{r}}{\partial t} \times \frac{\partial\mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 2\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos t, 2\sin t, 0\rangle.$$

- This is indeed an outward-pointing normal vector, since it is the vector pointing from $(0,0,s)$ to the point $\mathbf{r}(s,t) = (2\cos t, 2\sin t, s)$ on the surface.

Flux Across Surfaces, X

Example: Consider the vector field $\mathbf{F} = \left\langle xz^2, yz^2, x^3 e^y \right\rangle$ on the portion of the cylinder $x^2 + y^2 = 4$ between $z = -1$ and $z = 1$.

3. Set up and evaluate the flux of $\mathbf{F}$ across S.

Flux Across Surfaces, X

Example: Consider the vector field $\mathbf{F} = \left\langle xz^2, yz^2, x^3 e^y \right\rangle$ on the portion of the cylinder $x^2 + y^2 = 4$ between $z = -1$ and $z = 1$.

3. Set up and evaluate the flux of $\mathbf{F}$ across S.

- Since $x = 2\cos t$, $y = 2\sin t$, and $z = s$, we see $\mathbf{F} = \langle xz^2, yz^2, x^3 e^y \rangle = \langle 2s^2 \cos t, 2s^2 \sin t, (2\cos t)^3 e^{2\sin t} \rangle$.
- Then $\mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} \right) = \left\langle 2s^2 \cos t, 2s^2 \sin t, (2\cos t)^3 e^{2\sin t} \right\rangle \cdot \langle 2\cos t,\ 2\sin t,\ 0 \rangle = 4s^2 \cos^2 t + 4s^2 \sin^2 t = 4s^2$.
- The flux integral is thus $\int_0^{2\pi} \int_{-1}^{1} 4s^2 \, ds \, dt = \int_0^{2\pi} \frac{4}{3} s^3 \Big|_{s=-1}^{1} dt = \int_0^{2\pi} \frac{8}{3} \, dt = \frac{16\pi}{3}$.

Flux Across Surfaces, XI

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$.

1. Find a parametrization for the sphere.
2. Find the outward normal vector to the sphere.
3. Set up and evaluate the flux of $\mathbf{F}$ across S.

Flux Across Surfaces, XI

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$.

1. Find a parametrization for the sphere.
2. Find the outward normal vector to the sphere.
3. Set up and evaluate the flux of $\mathbf{F}$ across S.

- Using spherical coordinates, we can parametrize the hemisphere as $\mathbf{r}(s,t) = \langle 3 \sin s \cos t, 3 \sin s \sin t, 3 \cos s \rangle$ for $0 \le s \le \pi/2$ and $0 \le t \le 2\pi$.

Flux Across Surfaces, XII

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$.

2. Find the outward normal vector to the sphere.

- We have $\mathbf{r}(s,t) = \langle 3\sin s \cos t, 3\sin s \sin t, 3\cos s \rangle$.

Flux Across Surfaces, XII

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$.

2. Find the outward normal vector to the sphere.

- We have $\mathbf{r}(s,t) = \langle 3\sin s \cos t, 3\sin s \sin t, 3\cos s \rangle$.
- Then $\partial \mathbf{r}/\partial t = \langle -3\sin s \sin t, 3\sin s \cos t, 0 \rangle$ and $\partial \mathbf{r}/\partial s = \langle 3\cos s \cos t, 3\cos s \sin t, -3\sin s \rangle$, so
$$\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin s \sin t & 3\sin s \cos t & 0 \\ 3\cos s \cos t & 3\cos s \sin t & -3\sin s \end{vmatrix} =$$
$$\left\langle -9\sin^2 s \cos t, -9\sin^2 s \sin t, -9\sin s \cos s \right\rangle.$$
- However, this is actually an inward-pointing normal vector, since it is $-3\sin s$ times the position vector $\mathbf{r}(s,t)$.
- So we must scale it by -1 to get the actual outward normal, $\left\langle 9\sin^2 s \cos t, 9\sin^2 s \sin t, 9\sin s \cos s \right\rangle$.

Flux Across Surfaces, XIII

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$.

3. Set up and evaluate the flux of $\mathbf{F}$ across S.

Flux Across Surfaces, XIII

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$.

3. Set up and evaluate the flux of $\mathbf{F}$ across S.

- We have $\mathbf{r}(s,t) = \langle 3\sin s\cos t, 3\sin s\sin t, 3\cos s\rangle$ for $0 \le s \le \pi/2$ and $0 \le t \le 2\pi$.
- So $\mathbf{F} = \langle 6\sin s\cos t, 6\sin s\sin t, 6\cos s\rangle$.
- Then $\mathbf{F} \cdot -\left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial s}\right) = \langle 6\sin s\cos t, 6\sin s\sin t, 6\cos s\rangle \cdot \left\langle 9\sin^2 s\cos t, 9\sin^2 s\sin t, 9\sin s\cos s\right\rangle =$ $54\sin^3 s\cos^2 t + 54\sin^3 s\sin^2 t + 54\sin s\cos^2 s = 54\sin s$.
- The flux of $\mathbf{F}$ across S is therefore $\int_0^{2\pi}\int_0^{\pi/2} 54\sin s\,ds\,dt = \int_0^{2\pi} -54\cos s\Big|_{s=0}^{\pi/2} dt = \int_0^{2\pi} 54\,dt = 108\pi$.

Flux Across Surfaces, XIV

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$ using the implicit surface formula.

Flux Across Surfaces, XIV

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$ using the implicit surface formula.

- If we use the implicit surface formula instead, then the flux is given by $\iint_R \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx$, where $g(x, y, z) = x^2 + y^2 + z^2$.

Flux Across Surfaces, XIV

Example: Find the outward flux of the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ through the top half of the sphere $x^2 + y^2 + z^2 = 9$ using the implicit surface formula.

- If we use the implicit surface formula instead, then the flux is given by $\iint_R \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx$, where $g(x, y, z) = x^2 + y^2 + z^2$.
- We have $\nabla g = \langle 2x, 2y, 2z \rangle$, and the region R is the interior of the circle $x^2 + y^2 = 9$. Therefore, the flux integral is $\iint_R \frac{\langle 2x, 2y, 2z \rangle \cdot \langle 2x, 2y, 2z \rangle}{2z}\, dA = \iint_R \frac{36}{2\sqrt{9 - x^2 - y^2}}\, dy\, dx$.
- Switching to polar coordinates yields the explicit integral

$$\int_0^{2\pi} \int_0^3 \frac{36}{2\sqrt{9 - r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} -18\sqrt{9 - r^2}\Big|_{r=0}^{3}\, d\theta$$

$$= \int_0^{2\pi} 54\, d\theta = 108\pi.$$

Flux Across Surfaces, XV

Example: Find the flux of the vector field $\mathbf{F} = \langle 2xz, 2xy, 2z \rangle$ through the portion S of the surface $z = x^2 + y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, with upward orientation.

Flux Across Surfaces, XV

Example: Find the flux of the vector field $\mathbf{F} = \langle 2xz, 2xy, 2z \rangle$ through the portion S of the surface $z = x^2 + y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, with upward orientation.

- The description of the portion of the surface suggests using cylindrical coordinates to write down a parametrization.
- In cylindrical, the surface is $z = r^2$, and the portion we want has $1 \le r \le 2$.
- Thus, we get a parametrization $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$ with $1 \le r \le 2$ and $0 \le \theta \le 2\pi$.

Flux Across Surfaces, XVI

Example: Find the flux of the vector field $\mathbf{F} = \langle 2xz, 2xy, 2z \rangle$ through the portion S of the surface $z = x^2 + y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, with upward orientation.

- If $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$,

Flux Across Surfaces, XVI

Example: Find the flux of the vector field $\mathbf{F} = \langle 2xz, 2xy, 2z \rangle$ through the portion S of the surface $z = x^2 + y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, with upward orientation.

- If $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$, then $\partial\mathbf{r}/\partial r = \langle \cos\theta, \sin\theta, 2r \rangle$ and $\partial\mathbf{r}/\partial\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$.
- So then we get $\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$
$= \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle.$
- This normal vector does point upward, as required, since the z-coordinate is positive.

Flux Across Surfaces, XVII

Example: Find the flux of the vector field $\mathbf{F} = \langle 2xz, 2yz, 2z \rangle$ through the portion S of the surface $z = x^2 + y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, with upward orientation.

Flux Across Surfaces, XVII

Example: Find the flux of the vector field $\mathbf{F} = \langle 2xz, 2yz, 2z \rangle$ through the portion S of the surface $z = x^2 + y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, with upward orientation.

- We have $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$ for $1 \le r \le 2$ and $0 \le \theta \le 2\pi$. Thus $\mathbf{F} = \left\langle 2r^3\cos\theta, 2r^3\sin\theta, 2r^2 \right\rangle$.
- Then $\mathbf{F} \cdot \left(\dfrac{\partial \mathbf{r}}{\partial t} \times \dfrac{\partial \mathbf{r}}{\partial s} \right) =$ $\left\langle 2r^3\cos\theta, 2r^3\sin\theta, 2r^2 \right\rangle \cdot \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle$ $= -4r^5\cos^2\theta - 4r^5\sin^2\theta + 2r^3 = 2r^3 - 4r^5$.
- The flux of $\mathbf{F}$ across S is therefore $\displaystyle\int_0^{2\pi} \int_1^2 (2r^3 - 4r^5)\, dr\, d\theta = \int_0^{2\pi} -\frac{69}{2}\, d\theta = -69\pi.$

Flux Across Surfaces, XVIII

Example: Compute the flux of $\mathbf{F} = \left\langle -xz, \, -yz, \, x^2 + y^2 \right\rangle$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 6$, with upward orientation.

Flux Across Surfaces, XVIII

Example: Compute the flux of $\mathbf{F} = \left\langle -xz, -yz, x^2 + y^2 \right\rangle$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 6$, with upward orientation.

- In cylindrical the cone is $z = r$, so using parameters r, θ we get the parametrization $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ for $0 \le r \le \sqrt{6}$, $0 \le \theta \le 2\pi$.

Flux Across Surfaces, XVIII

Example: Compute the flux of $\mathbf{F} = \left\langle -xz, -yz, x^2 + y^2 \right\rangle$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 6$, with upward orientation.

- In cylindrical the cone is $z = r$, so using parameters r, θ we get the parametrization $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ for $0 \le r \le \sqrt{6}$, $0 \le \theta \le 2\pi$.
- Then $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r\cos\theta, -r\sin\theta, r\rangle$, which has upward orientation since the z-coordinate is positive.
- Then $\mathbf{F} = \left\langle -r^2\cos\theta, -r^2\sin\theta, r^2 \right\rangle$ and so $\mathbf{F} \cdot \mathbf{n} = \left\langle -r^2\cos\theta, -r^2\sin\theta, r^2 \right\rangle \cdot \langle -r\cos\theta, -r\sin\theta, r\rangle = 2r^3$.
- Thus, the flux is $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{6}} 2r^3\, dr\, d\theta = \int_0^{2\pi} 18\, d\theta = 36\pi$.

Summary

We introduced flux across a surface.

We discussed how to calculate flux across parametric surfaces and how to calculate flux across implicit surfaces.

Next lecture: Conservative vector fields and potential functions.