

Math 2321 (Multivariable Calculus)

Lecture #28 of 37 ~ March 29, 2021

Vector Fields, Work, Circulation, and Flux

- Vector Fields
- Circulation and Work Integrals in 2-Space and 3-Space
- Flux Across Curves

This material represents §4.3.1-4.3.2 from the course notes.

Vector Fields, I

We now discuss vector fields and their applications. A vector field is a function that assigns a vector to each point in space.

- Thus, a vector field in 2 dimensions is a vector-valued function of the form $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$, while a vector field in 3 dimensions is a vector-valued function of the form $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ for some functions P, Q, R .
- Example: Two vector fields in the plane are $\mathbf{F}(x, y) = \langle x^2 + y, xy \rangle$ and $\mathbf{G}(x, y) = \langle -y, x \rangle$.
- Another class of vector fields we have already encountered is the vector field associated to the gradient of a function $f(x, y)$ or $f(x, y, z)$: for example, if $f(x, y) = x^2 + xy$, then $\nabla f(x, y) = \langle 2x + y, x \rangle$ is a vector field in the plane.

Vector Fields, II

Vector fields can be used to model fluid flow.

- Specifically, if we have a fluid flowing in the plane, then we obtain a vector field $\mathbf{F}(x, y)$ measuring the velocity of the fluid at the point (x, y) .
- Likewise, if we have a fluid flowing in space, then we obtain a vector field $\mathbf{F}(x, y, z)$ measuring the velocity of the fluid at the point (x, y, z) .
- Motivating question #1: Given a path, how can we measure how much fluid travels along the path?
- Motivating question #2: Given a region, how can we measure how much fluid flows in or out of the region?

Vector Fields, III

Vector fields can also be used to model the actions of forces (sometimes they are called “force fields” in this situation).

- If we have a physical phenomenon that imparts forces to objects, we obtain a vector field \mathbf{F} by measuring the force imparted at a particular point. (Note that forces are vectors, since they have magnitude and direction.)
- Common examples include magnetic fields, electric fields, or gravitational fields.
- Motivating question: If a particle travels along a path through one of these “force fields”, how can we measure the work done by the field on the particle?

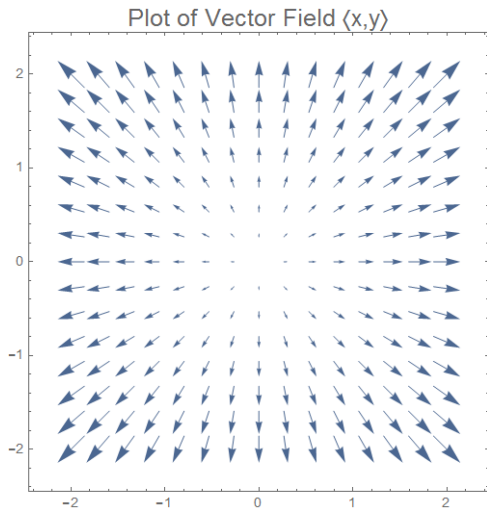
Vector Fields, IV

To represent a vector field visually, we choose some (nice) collection of points (generally in a grid) and draw the vectors corresponding to those points as arrows pointing in the appropriate direction and with the appropriate length.

- These plots tend to be rather tedious to produce by hand, since it requires computing and then drawing a large number of vectors at a scale that captures the relative magnitudes of the vectors, but does not become too cluttered.
- Naturally, we will prefer to use a computer to draw vector field plots.

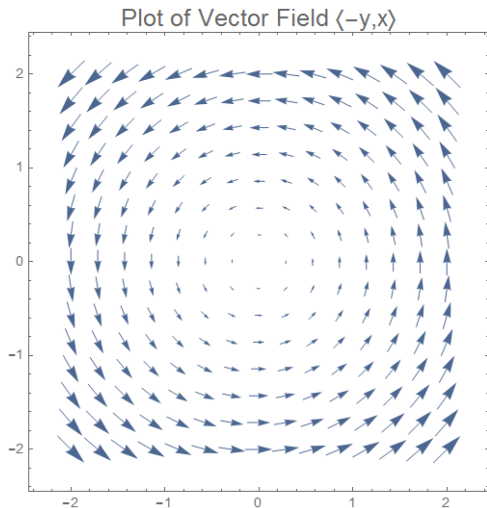
Vector Fields, V

Here is a plot of $\mathbf{F}(x, y) = \langle x, y \rangle$:



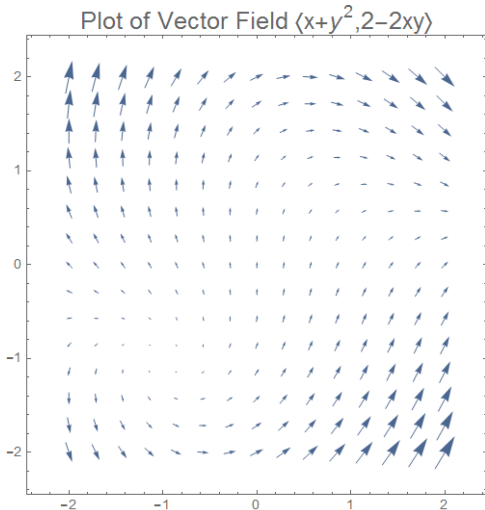
Vector Fields, VI

Here is a plot of $\mathbf{F}(x, y) = \langle -y, x \rangle$:



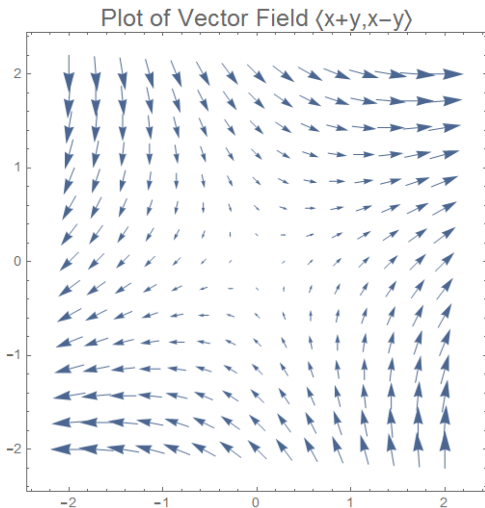
Vector Fields, VII

Here is a plot of $\mathbf{F}(x, y) = \langle x + y^2, 2 - 2xy \rangle$:



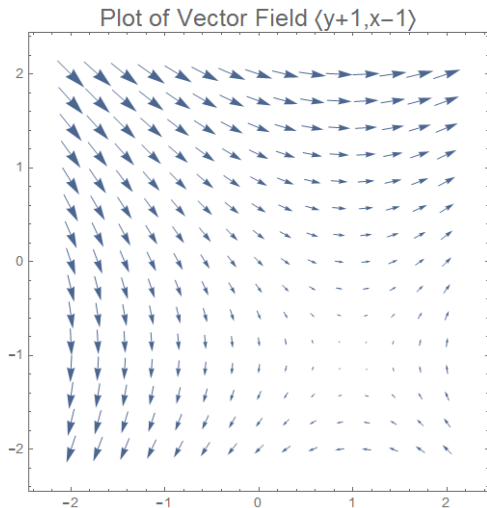
Vector Fields, VIII

Here is a plot of $\mathbf{F}(x, y) = \langle x + y, x - y \rangle$:



Vector Fields, IX

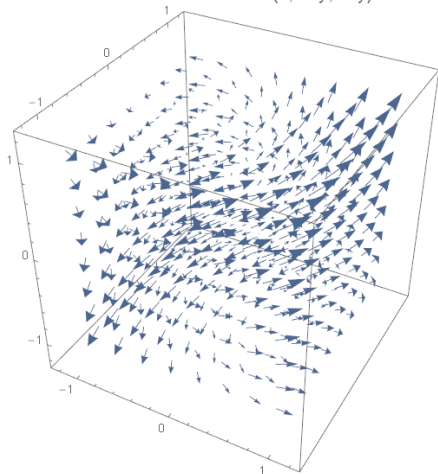
Here is a plot of $\mathbf{F}(x, y) = \langle y + 1, x - 1 \rangle$:



Vector Fields, X

We can also produce these plots in 3D, but they are usually very cluttered. Here is one for $\mathbf{F}(x, y, z) = \langle x, z - y, x + y \rangle$:

Plot of Vector Field $\langle x, z - y, x + y \rangle$



Work, I

Let's now examine the behavior of vector fields representing forces.

- If \mathbf{F} represents the force imparted to a particle at a given position, we would like to calculate the total work done on the particle as it travels along a path.
- To visualize this, imagine you are riding a bicycle on a windy day. The wind can either be helping you (if it is a tailwind, pushing you from behind) or hindering you (if it is a headwind, pushing you from in front).
- The total amount of help, or hindrance, the wind provides will then depend on how much it is pushing you in the direction you are already moving.

Work, II

If we have a parametrization $\mathbf{r}(t)$ of the path, then we are seeking to measure “how much” of the force vector \mathbf{F} is in the direction of the path.

Work, II

If we have a parametrization $\mathbf{r}(t)$ of the path, then we are seeking to measure “how much” of the force vector \mathbf{F} is in the direction of the path.

- This is the same as asking how much of the force vector \mathbf{F} is in the direction tangent to the path.
- The tangent direction to the path is given by the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$.
- We therefore want to know how much of \mathbf{F} is in the direction of \mathbf{T} .
- From our discussion of dot products and vector projections, this is precisely what the dot product $\mathbf{F} \cdot \mathbf{T}$ measures.

Work, III

Our analysis indicates that the total work that a vector field \mathbf{F} does on a particle traveling on a path $\mathbf{r}(t)$ is given by integrating the dot product $\mathbf{F} \cdot \mathbf{T}$ along the path.

Definition

The work performed on a particle by a vector field \mathbf{F} as the particle travels along a curve C is $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

In order to evaluate the integral as written, we would need to parametrize the curve C , find the unit tangent vector $\mathbf{T}(t)$ to the curve, and then integrate the dot product $\mathbf{F}(x(t), y(t)) \cdot \mathbf{T}(t)$ along the curve. Rather than setting up the problem this way, we will investigate a more efficient approach.

Work, IV

So suppose that $\mathbf{F}(x, y) = \langle P, Q \rangle$, where P and Q are functions of x and y and C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$.

- Then $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle dx/dt, dy/dt \rangle}{\|\mathbf{v}(t)\|}$.
- Thus, $\mathbf{F} \cdot \mathbf{T} = \frac{\langle P, Q \rangle \cdot \langle dx/dt, dy/dt \rangle}{\|\mathbf{v}(t)\|} = \frac{P \frac{dx}{dt} + Q \frac{dy}{dt}}{\|\mathbf{v}(t)\|}$.
- Therefore, the work integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T} \, ds &= \int_a^b \frac{P \frac{dx}{dt} + Q \frac{dy}{dt}}{\|\mathbf{v}(t)\|} \|\mathbf{v}(t)\| \, dt \\ &= \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} \right] dt. \end{aligned}$$

Work, \mathcal{V}

Thus, to summarize, the work done by a vector field $\mathbf{F}(x, y)$ on a particle traveling along a curve C in the plane is given by

$$\int_C P dx + Q dy = \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} \right] dt.$$

- We can also pose essentially the same definition for a curve in 3-space, and we obtain an analogous formula.
- Explicitly, if $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$, then the work done by \mathbf{F} on a particle traveling along a curve C in 3-space is

$$\int_C P dx + Q dy + R dz = \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right] dt.$$

Work, VI

Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle y, x^2y \rangle$ on a particle traveling the curve $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \leq t \leq 2$.

Work, VI

Example: Find the work done by the vector field $\mathbf{F}(x, y) = \langle y, x^2y \rangle$ on a particle traveling the curve $\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \leq t \leq 2$.

- We need to compute $\int_C P dx + Q dy$.
- On this curve, $x = t$ and $y = t^2$.
- So $P = y = t^2$ and $Q = x^2y = t^4$.
- Also, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2t$.

- Therefore, the work is $\int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt =$
 $\int_0^2 [t^2 \cdot 1 + t^4 \cdot 2t] dt = \int_0^2 (t^2 + 2t^5) dt = 24.$

Work, VII

Example: Find the work done by the vector field

$\mathbf{F}(x, y, z) = \langle 2x + z, yz, xy \rangle$ on a particle traveling along the path
 $\mathbf{r}(t) = \langle t, t^2, 2t \rangle$ from $t = 0$ to $t = 1$.

Work, VII

Example: Find the work done by the vector field

$\mathbf{F}(x, y, z) = \langle 2x + z, yz, xy \rangle$ on a particle traveling along the path $\mathbf{r}(t) = \langle t, t^2, 2t \rangle$ from $t = 0$ to $t = 1$.

- We need to compute $\int_C P dx + Q dy + R dz$.
- On this curve, $x = t$, $y = t^2$, $z = 2t$.
- We have $P = 2x + z = 4t$, $Q = yz = 2t^3$, and $R = xy = t^3$.
Also, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$, and $\frac{dz}{dt} = 2$.

- Therefore, the work is $\int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt =$
 $\int_0^1 [(4t)(1) + (2t^3)(2t) + (t^3)(2)] dt =$
 $\int_0^1 (4t + 4t^4 + 2t^3) dt = \frac{33}{10}.$

Work, VIII

We will also mention another different form for the work integral

$$\text{Work} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right] dt.$$

- If we define the “vector differential” $d\mathbf{r} = \langle dx, dy, dz \rangle$, then we can think of the work integral as being the integral of the formal dot product $\mathbf{F} \cdot d\mathbf{r} = P \, dx + Q \, dy + R \, dz$.
- Thus, the work integral is often also written in the form

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

and is sometimes referred to as “the integral of the vector field \mathbf{F} on the curve C ”.

- Note also the units of work: if the force field \mathbf{F} is measured in newtons, with distances in meters, then the units of $\int_C \mathbf{F} \cdot d\mathbf{r}$ are newton-meters (i.e., joules).

Circulation and Flux, I

Now we will study the behavior of vector fields representing fluid flow.

- Imagine again that you are riding a bicycle on a windy day, and we want to measure how much air flows along the path you take.
- We can see that air flows along your path whenever its velocity is in the same direction as your path.
- Therefore, just like the work integral, this is measuring how much the vector field \mathbf{F} points in the same direction as the tangent vector \mathbf{T} to your path.

Circulation and Flux, II

The resulting quantity, measuring how much the fluid flows along a given path, is called circulation:

Definition

If \mathbf{F} is a vector field representing the velocity of a fluid flowing through space, then the (counterclockwise) circulation (or flow) of the vector field \mathbf{F} along the curve C is defined to be

$$\text{Circulation} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds,$$

where \mathbf{T} is the unit tangent vector to the curve C .

The circulation measures the total amount of fluid flowing along the curve.

Circulation and Flux, III

The circulation integral has the same form as the work integral, so we can evaluate it in exactly the same way.

- In the plane, if $\mathbf{F} = \langle P, Q \rangle$ and C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$, the circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} \right] dt.$$

- In 3-space, if $\mathbf{F} = \langle P, Q, R \rangle$ and C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$, the circulation is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right] dt.$$

Although the circulation integral has the same form as the work integral, the interpretation and the units are different.

- Here, \mathbf{F} is now a velocity field, with units of meters per second: then the units of $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ are square meters per second. The circulation integral represents a rate of fluid flow.

Circulation and Flux, IV

Example: Find the circulation of $\mathbf{F}(x, y) = \langle -y, x \rangle$ around a path that winds once counterclockwise around the unit circle.

Circulation and Flux, IV

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- We need to compute $\int_C P dx + Q dy$.
- We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$.

Circulation and Flux, IV

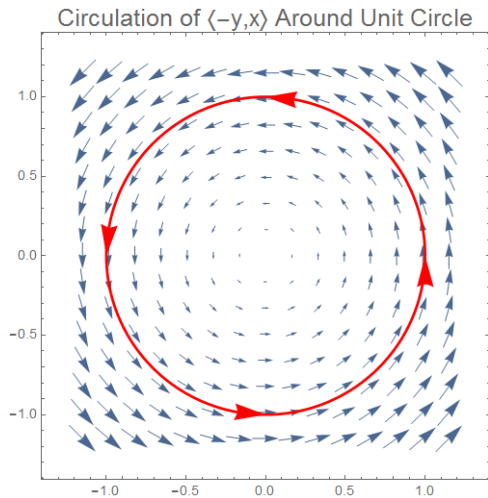
Example: Find the circulation of $\mathbf{F}(x, y) = \langle -y, x \rangle$ around a path that winds once counterclockwise around the unit circle.

- We need to compute $\int_C P dx + Q dy$.
- We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$.
- Thus, $P = -y = -\sin t$ and $Q = x = \cos t$, and also $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$.

- So, the circulation is $\int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt = \int_0^{2\pi} ((-\sin t)(-\sin t) + (\cos t)(\cos t)) dt = \int_0^{2\pi} 1 dt = 2\pi$.

Circulation and Flux, ∇

Here is a picture of $\mathbf{F}(x, y) = \langle -y, x \rangle$ along with the unit circle:



Circulation and Flux, VI

Example: Find the circulation of $\mathbf{F}(x, y, z) = \langle 2xz, 2z^2, y \rangle$ along the line segment from $(0, 1, 0)$ to $(2, 2, 2)$.

Circulation and Flux, VI

Example: Find the circulation of $\mathbf{F}(x, y, z) = \langle 2xz, 2z^2, y \rangle$ along the line segment from $(0, 1, 0)$ to $(2, 2, 2)$.

- We need to compute $\int_C P dx + Q dy + R dz$.
- We can parametrize the path as $x = 2t$, $y = 1 + t$, $z = 2t$ for $0 \leq t \leq 1$.

Circulation and Flux, VI

Example: Find the circulation of $\mathbf{F}(x, y, z) = \langle 2xz, 2z^2, y \rangle$ along the line segment from $(0, 1, 0)$ to $(2, 2, 2)$.

- We need to compute $\int_C P dx + Q dy + R dz$.
- We can parametrize the path as $x = 2t, y = 1 + t, z = 2t$ for $0 \leq t \leq 1$.
- Then $dx/dt = 2, dy/dt = 1, dz/dt = 2$.
- Also, $P = 2xz = 8t^2, Q = 2z^2 = 8t^2, R = y = 1 + t$.
- So, the circulation is

$$\begin{aligned} \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt &= \int_0^1 (8t^2 \cdot 2 + 8t^2 \cdot 1 + (1 + t) \cdot 2) dt \\ &= \int_0^1 (24t^2 + 2t + 2) dt = 11. \end{aligned}$$

Circulation and Flux, VII

We can also measure the amount of fluid that flows across the path, rather than along the path.

Circulation and Flux, VII

We can also measure the amount of fluid that flows across the path, rather than along the path.

- If you are once again riding your bicycle on a windy day, what we now want to measure is how much the wind is pushing you off course.
- Equivalently, we want to measure how much of the vector field \mathbf{F} points in the direction perpendicular to the curve.
- In the plane, the unit normal vector \mathbf{N} is perpendicular to the unit tangent vector \mathbf{T} , and so \mathbf{N} is the direction perpendicular to the curve.
- Thus, to measure the amount of fluid flowing across the curve, we want to integrate $\mathbf{F} \cdot \mathbf{N}$ along the curve.

Circulation and Flux, VIII

The resulting quantity, measuring how much the fluid flows across a given plane curve, is called flux:

Definition

If \mathbf{F} is a vector field representing the velocity of a fluid flowing through space, then the (normal) flux of the vector field \mathbf{F} across the curve C is

$$\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{N} \, ds$$

where \mathbf{N} is the unit normal vector to the curve.

The flux measures the total amount of fluid flowing across the curve.

Just as with the circulation integral, we could set this up as a line integral by computing \mathbf{N} directly. But this is quite messy.

Circulation and Flux, IX

So suppose that $\mathbf{F}(x, y) = \langle P, Q \rangle$, where P and Q are functions of x and y and C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$.

- Then $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle dx/dt, dy/dt \rangle}{\|\mathbf{v}(t)\|}$. Some algebra then gives $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\langle dy/dt, -dx/dt \rangle}{\|\mathbf{v}(t)\|}$.
- Then $\mathbf{F} \cdot \mathbf{N} = \frac{\langle P, Q \rangle \cdot \langle dy/dt, -dx/dt \rangle}{\|\mathbf{v}(t)\|} = \frac{P \frac{dy}{dt} - Q \frac{dx}{dt}}{\|\mathbf{v}(t)\|}$.
- Therefore, the flux integral is given explicitly by

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{N} \, ds &= \int_a^b \frac{P \frac{dy}{dt} - Q \frac{dx}{dt}}{\|\mathbf{v}(t)\|} \|\mathbf{v}(t)\| \, dt \\ &= \int_a^b \left[-Q \frac{dx}{dt} + P \frac{dy}{dt} \right] dt. \end{aligned}$$

Circulation and Flux, X

Thus, to summarize, the flux of the vector field $\mathbf{F}(x, y)$ across the curve C in the plane is given by

$$\text{Flux} = \int_C -Q dx + P dy = \int_a^b \left[-Q \frac{dx}{dt} + P \frac{dy}{dt} \right] dt.$$

- Unlike with circulation, however, the 3-dimensional version of this formula is quite a bit different.
- The reason is that the flux integral is intended to measure fluid flow across a membrane.
- In 2-space, a membrane will be a curve, but in 3-space, a membrane will be a surface, and so flux integrals in 3-space are surface integrals. (We will discuss them next class.)

Circulation and Flux, XI

Example: Find the flux of the vector field $\mathbf{G}(x, y) = \langle x, y \rangle$ across a path that winds once counterclockwise around the unit circle.

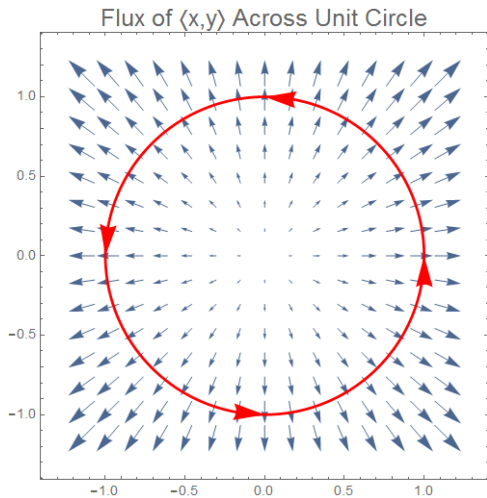
Circulation and Flux, XI

Example: Find the flux of the vector field $\mathbf{G}(x, y) = \langle x, y \rangle$ across a path that winds once counterclockwise around the unit circle.

- We need to compute $\int_C -Q dx + P dy$.
- We can parametrize the path as $x = \cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$.
- Then $dx/dt = -\sin t$ and $dy/dt = \cos t$.
- Also, $P = x = \cos t$ and $Q = y = \sin t$.
- Therefore, the flux integral is $\int_a^b \left(-Q \frac{dx}{dt} + P \frac{dy}{dt} \right) dt = \int_0^{2\pi} (-(\sin t)(-\sin t) + (\cos t)(\cos t)) dt = \int_0^{2\pi} 1 dt = 2\pi$.

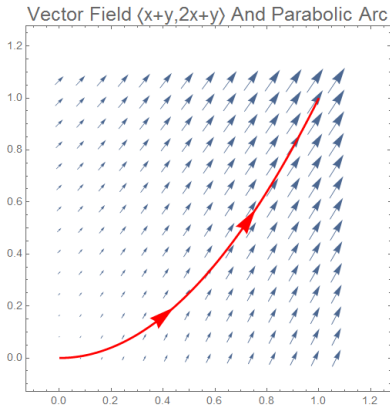
Circulation and Flux, XII

Here is a picture of $\mathbf{F}(x, y) = \langle x, y \rangle$ along with the unit circle:



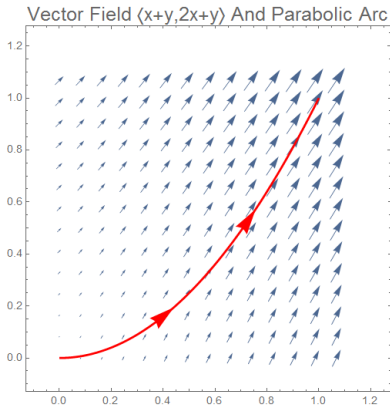
Circulation and Flux, XIII

Example: For the vector field $\mathbf{F}(x, y) = \langle x + y, 2x + y \rangle$, find the flux across, and circulation along, the parabolic arc $\mathbf{r}(t) = \langle t, t^2 \rangle$ between $(0, 0)$ and $(1, 1)$.



Circulation and Flux, XIII

Example: For the vector field $\mathbf{F}(x, y) = \langle x + y, 2x + y \rangle$, find the flux across, and circulation along, the parabolic arc $\mathbf{r}(t) = \langle t, t^2 \rangle$ between $(0, 0)$ and $(1, 1)$.



- Here, we can see that the vector field primarily points along the curve most of the way, which suggests that the circulation should be fairly large and positive.
- The field does not flow substantially across the curve anywhere, so the flux should be relatively small.

Circulation and Flux, XIV

Example: For the vector field $\mathbf{F}(x, y) = \langle x + y, 2x + y \rangle$, find the flux across, and circulation along, the portion of the curve $\mathbf{r}(t) = \langle t, t^2 \rangle$ between $(0, 0)$ and $(1, 1)$.

Circulation and Flux, XIV

Example: For the vector field $\mathbf{F}(x, y) = \langle x + y, 2x + y \rangle$, find the flux across, and circulation along, the portion of the curve $\mathbf{r}(t) = \langle t, t^2 \rangle$ between $(0, 0)$ and $(1, 1)$.

- We have $x = t$ and $y = t^2$, so $dx/dt = 1$ and $dy/dt = 2t$.

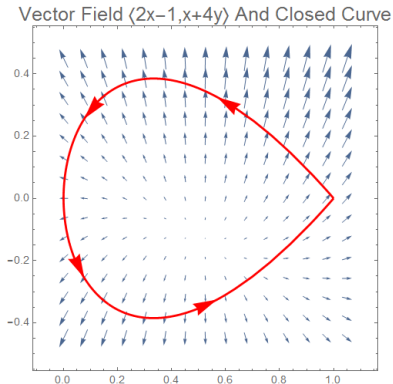
- Also, $P = x + y = t^2 + t$ and $Q = 2x + y = 2t^2 + t$.

- The circulation is $\int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt$
$$= \int_0^1 [(t^2 + t) \cdot 1 + (2t^2 + t) \cdot 2t] dt = \int_0^1 [4t^3 + 3t^2 + t] dt = \frac{5}{2}.$$

- The flux is $\int_a^b \left(-Q \frac{dx}{dt} + P \frac{dy}{dt} \right) dt$
$$= \int_0^1 [-(2t^2 + t) \cdot 1 + (t^2 + t) \cdot 2t] dt = \int_0^1 [2t^3 - 2t] dt = -\frac{1}{2}.$$

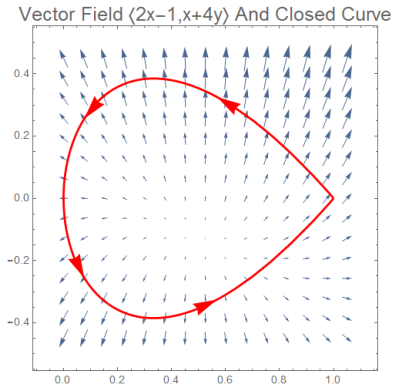
Circulation and Flux, XV

Example: For $\mathbf{F}(x, y) = \langle 2x - 1, 4y \rangle$, find the flux across, and circulation around, the curve $\mathbf{r}(t) = \langle t^2, t^3 - t \rangle$ for $-1 \leq t \leq 1$.



Circulation and Flux, XV

Example: For $\mathbf{F}(x, y) = \langle 2x - 1, 4y \rangle$, find the flux across, and circulation around, the curve $\mathbf{r}(t) = \langle t^2, t^3 - t \rangle$ for $-1 \leq t \leq 1$.



- The field flows outward across the curve most everywhere, so we would expect the flux to be fairly large and positive.
- It also flows a bit along the curve on the bottom right, so we would also expect the circulation to be positive, but not as large.

Circulation and Flux, XVI

Example: For $\mathbf{F}(x, y) = \langle 2x - 1, x + 4y \rangle$, find the flux across, and circulation around, the curve $\mathbf{r}(t) = \langle t^2, t^3 - t \rangle$ for $-1 \leq t \leq 1$.

- We have $x = t^2$ and $y = t^3 - t$.
- Thus $dx/dt = 2t$ and $dy/dt = 3t^2 - 1$.
- Also, $P = 2x - 1 = 2t^2 - 1$ and $Q = x + 4y = 4t^3 + t^2 - 4t$.

- The circulation is
$$\int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt$$
$$= \int_{-1}^1 [(2t^2 - 1)(2t) + (4t^3 + t^2 - 4t)(3t^2 - 1)] dt = \frac{8}{15}.$$

- The flux is
$$\int_a^b \left(-Q \frac{dx}{dt} + P \frac{dy}{dt} \right) dt$$
$$= \int_{-1}^1 [-(4t^3 + t^2 - 4t)(2t) + (2t^2 - 1)(3t^2 - 1)] dt = \frac{16}{5}.$$

Summary

We introduced vector fields and ways to represent them algebraically and geometrically.

We introduced work, circulation, and flux integrals (in the plane), and described how to calculate them.

Next lecture: Flux across surfaces.