Math 2321 (Multivariable Calculus)

Lecture #27 of 37 $\sim$ March 25, 2021

Surface Integrals

- Surface Integrals
- Computing Surface Integrals
- Applications of Surface Integrals

This material represents §4.2.2 from the course notes.

Surface Integrals, I

Now that we have learned how to parametrize various kinds of surfaces, we can discuss surface integrals.

- One motivating application is to calculate the surface area of a given surface.
- Another application is to find the average value of a function on a surface, or to compute the mass of a surface region with variable density.
- In a similar way to how we computed line integrals using (single) integrals, we will be able to compute surface integrals as double integrals.

Surface Integrals, II

As with all the other types of integrals, we start by defining surface integrals in terms of Riemann sums.

Definition

For a parametric surface S defined in terms of parameters s and t, a partition of S into n pieces is a list of disjoint subregions inside S, where the kth subregion corresponds to $s_k \leq s \leq s_k'$, $t_k \leq t \leq t_k'$, and has surface area $\Delta\sigma_k$.
The norm of the partition P is the largest number among the areas of the rectangles in P.
For a continuous function $f(x, y, z)$ and a partition P a partition of the surface S, we define the Riemann sum of $f(x, y, z)$ on R corresponding to P to be $RS_P(f) = \sum_{k=1}^{n} f(\mathbf{r}(s_k, t_k))\, \Delta\sigma_k$.

Surface Integrals, III

And here is the definition of the surface integral, which is essentially the limit of the Riemann sums as we divide the surface into arbitrarily small pieces:

Definition

For a function $f(x, y, z)$, we define the surface integral of f on S, denoted $\iint_S f(x, y, z)\, d\sigma$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\mathrm{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.

Remark: It can be proven (with significant effort) that, if $f(x, y, z)$ is continuous, then a value of L satisfying the hypotheses actually does exist.

Surface Integrals, IV

As with all of the other types of integrals, surface integrals possess some formal properties. For any continuous functions f and g, and any constant C, we have the following:

1. Integral of constant: $\iint_S C\, d\sigma = C \cdot \text{Area}(S)$.
2. Constant multiple of a function: $\iint_S C f\, d\sigma = C \cdot \iint_S f\, d\sigma$.
3. Addition of functions: $\iint_S f\, d\sigma + \iint_S g\, d\sigma = \iint_S [f + g]\, d\sigma$.
4. Subtraction of functions: $\iint_S f\, d\sigma - \iint_S g\, d\sigma = \iint_S [f - g]\, d\sigma$.
5. Nonnegativity: if $f \geq 0$, then $\iint_S f\, d\sigma \geq 0$.
6. Union: If S_1 and S_2 don't overlap and have union S, then $\iint_{S_1} f\, d\sigma + \iint_{S_2} f\, d\sigma = \iint_S f\, d\sigma$.

Computing Surface Integrals, I

We were able to reduce line integral calculations to standard one-variable integrals. We can similarly reduce calculations of surface integrals to double integrals:

Proposition (Parametric Surface Integrals)

If $f(x, y, z)$ is continuous on the surface S which is parametrized as $\mathbf{r}(s, t) = \langle x(s, t),\ y(s, t),\ z(s, t)\rangle$, where S is described by a region R in st-coordinates, then the surface integral of f on S is

$$\iint_S f\, d\sigma = \iint_R f(x(s, t),\ y(s, t),\ z(s, t)) \left|\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|\right|\, dt\, ds.$$

The idea is just to write down the Riemann sums for the surface integral and recognize them also as Riemann sums for a double integral.

Computing Surface Integrals, II

Consider $\displaystyle\iint_S f\,d\sigma = \iint_R f(x(s,t),\,y(s,t),\,z(s,t)) \left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| dt\,ds.$

- The only non-obvious part is why the differential of surface area $d\sigma$ is $\displaystyle\left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| dt\,ds$.

Computing Surface Integrals, II

Consider $\displaystyle\iint_S f\,d\sigma = \iint_R f(x(s,t),\, y(s,t),\, z(s,t)) \left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| \,dt\,ds.$

- The only non-obvious part is why the differential of surface area $d\sigma$ is $\left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right| \,dt\,ds$.
- The point is that $d\sigma$ arises from computing the area of a small patch in st-coordinates.
- When s changes slightly, the change in $\mathbf{r}$ is given by $\partial \mathbf{r}/\partial s$, and when t changes slightly, the change in $\mathbf{r}$ is given by $\partial \mathbf{r}/\partial t$.
- These two vectors form a small parallelogram that closely approximates the surface S, so the differential of surface area $d\sigma$ is roughly equal to the area of this parallelogram, which is $\left|\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|\right|$, times the differential $dt\,ds$.

Computing Surface Integrals, III

Example: Set up, and then evaluate, the surface integral of $g(x, y, z) = z$ on the surface with parametrization $\mathbf{r}(s, t) = \langle \sin(t), \cos(t), s + t \rangle$ for $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

Computing Surface Integrals, III

Example: Set up, and then evaluate, the surface integral of $g(x, y, z) = z$ on the surface with parametrization $\mathbf{r}(s, t) = \langle \sin(t), \cos(t), s + t \rangle$ for $0 \leq t \leq 2\pi$ and $0 \leq s \leq \pi$.

- On the surface, we have $z = s + t$ so $g(x, y, z) = z = s + t$.
- We have $\dfrac{\partial \mathbf{r}}{\partial s} = \langle 0, 0, 1 \rangle$ and $\dfrac{\partial \mathbf{r}}{\partial t} = \langle \cos(t), -\sin(t), 1 \rangle$, so
$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos(t) & -\sin(t) & 1 \end{vmatrix} = \langle \sin(t), \cos(t), 0 \rangle.$$
- So, we see $\left|\left| \dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t} \right|\right| = 1$. (All that work!)
- The integral is therefore $\displaystyle\int_0^{2\pi} \int_0^{\pi} (s + t)\, ds\, dt$.

Computing Surface Integrals, IV

Example: Set up, and then evaluate, the surface integral of $g(x, y, z) = z$ on the surface with parametrization $\mathbf{r}(s, t) = \langle \sin(t), \cos(t), s + t\rangle$ for $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

- Now we just evaluate it as an ordinary double integral:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}(s+t)\,ds\,dt &= \int_0^{2\pi}\left[\frac{s^2}{2}+st\right]\Big|_{s=0}^{\pi}dt \\
&= \int_0^{2\pi}\left[\frac{\pi^2}{2}+\pi t\right]\,dt \\
&= \left[\frac{\pi^2}{2}t+\frac{\pi}{2}t^2\right]\Big|_{t=0}^{2\pi} \\
&= 3\pi^3.
\end{aligned}$$

Computing Surface Integrals, V

Example: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \leq 6$.

1. Find a parametrization of S.
2. Set up $\iint_S g \, d\sigma$.
3. Evaluate $\iint_S g \, d\sigma$.

Computing Surface Integrals, V

Example: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \leq 6$.

1. Find a parametrization of S.
2. Set up $\iint_S g \, d\sigma$.
3. Evaluate $\iint_S g \, d\sigma$.

- Here we have to find the parametrization, and then set up and evaluate the integral.
- We can find the parametrization using cylindrical coordinates.
- Then we have to write down the function in terms of the parameters, and also compute the differential.

Computing Surface Integrals, VI

Example: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \leq 6$.

1. Find a parametrization of S.

Computing Surface Integrals, VI

<u>Example</u>: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \le 6$.

1. Find a parametrization of S.

- We can use cylindrical coordinates, since this portion of the cone will have a nice description in cylindrical.
- In cylindrical, the cone is $z = 3r$. Since z is determined, we will use the parameters r and θ.
- Since we want the portion with $z \le 6$, we need $0 \le r \le 2$.
- Then $x = r\cos\theta$, $y = r\sin\theta$, and $z = 3r$.
- Thus, we get a parametrization $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle$ with $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

Computing Surface Integrals, VII

Example: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \le 6$.

2. Set up $\iint_S g \, d\sigma$.

- Note $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle$ with $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Computing Surface Integrals, VII

<u>Example</u>: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \leq 6$.

2. Set up $\iint_S g \, d\sigma$.

- Note $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 3r\rangle$ with $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.
- Then $g(x, y, z) = r$ on this surface.
- Also, $\dfrac{\partial \mathbf{r}}{\partial r} = \langle \cos\theta, \sin\theta, 3\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$, so

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 3 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -3r\cos\theta, -3r\sin\theta, r\rangle.$$

- So, we see $\left|\left|\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta}\right|\right| = r\sqrt{10}$.
- The integral is therefore $\displaystyle\int_0^{2\pi}\int_0^2 r \cdot r\sqrt{10}\, dr\, d\theta$.

Computing Surface Integrals, VIII

Example: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \leq 6$.

3. Evaluate $\iint_S g \, d\sigma$.

- The integral is $\displaystyle\int_0^{2\pi} \int_0^2 r \cdot r\sqrt{10} \, dr \, d\theta$.

Computing Surface Integrals, VIII

Example: Consider the function $g(x, y, z) = \sqrt{x^2 + y^2}$ on the portion S of the cone $z = 3\sqrt{x^2 + y^2}$ where $z \le 6$.

3. Evaluate $\iint_S g \, d\sigma$.

- The integral is $\int_0^{2\pi} \int_0^2 r \cdot r\sqrt{10} \, dr \, d\theta$.
- Now we just evaluate it:

$$\begin{aligned} \int_0^{2\pi} \int_0^2 r \cdot r\sqrt{10} \, dr \, d\theta &= \int_0^{2\pi} \frac{1}{3} r^3 \sqrt{10} \Big|_{r=0}^{2} d\theta \\ &= \int_0^{2\pi} \frac{8}{3}\sqrt{10} \, d\theta = \frac{16}{3}\pi\sqrt{10}. \end{aligned}$$

Computing Surface Integrals, IX

<u>Example</u>: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

1. Find a parametrization of S.
2. Set up $\iint_S f \, d\sigma$.
3. Evaluate $\iint_S f \, d\sigma$.

Computing Surface Integrals, IX

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

1. Find a parametrization of S.
2. Set up $\iint_S f \, d\sigma$.
3. Evaluate $\iint_S f \, d\sigma$.

- Here we have to find the parametrization, and then set up and evaluate the integral.
- We can find the parametrization using spherical coordinates.
- Then we have to write down the function in terms of the parameters, and also compute the differential.

Computing Surface Integrals, X

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

1. Find a parametrization of S.

Computing Surface Integrals, X

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

1. Find a parametrization of S.

- We can use spherical coordinates.
- In spherical, the sphere is $\rho = 4$. Since ρ is determined, we will use the parameters θ and φ.
- Since we want the upper half, we want $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$.
- Then $x = 4 \sin \varphi \cos \theta$, $y = 4 \sin \varphi \sin \theta$, and $z = 4 \cos \varphi$.
- Thus, we get a parametrization $\mathbf{r}(\varphi, \theta) = \langle 4 \sin \varphi \cos \theta, 4 \sin \varphi \sin \theta, 4 \cos \varphi \rangle$ with $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$.

Computing Surface Integrals, XI

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

2. Set up $\iint_S f \, d\sigma$.

- Note $\mathbf{r}(\varphi, \theta) = \langle 4\sin\varphi\cos\theta, 4\sin\varphi\sin\theta, 4\cos\varphi\rangle$.

Computing Surface Integrals, XI

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

2. Set up $\iint_S f\, d\sigma$.
   - Note $\mathbf{r}(\varphi, \theta) = \langle 4 \sin\varphi \cos\theta, 4 \sin\varphi \sin\theta, 4\cos\varphi \rangle$.
   - Then $g(x, y, z) = 4\cos\varphi$ on this surface.
   - Also, $\partial \mathbf{r}/\partial\varphi = \langle 4\cos\varphi\cos\theta, 4\cos\varphi\sin\theta, -4\sin\varphi\rangle$ and $\partial\mathbf{r}/\partial\theta = \langle -4\sin\varphi\sin\theta, 4\sin\varphi\cos\theta, 0\rangle$, so
   $$\frac{\partial \mathbf{r}}{\partial\varphi} \times \frac{\partial \mathbf{r}}{\partial\theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos\varphi\cos\theta & 4\cos\varphi\sin\theta & -4\sin\varphi \\ -4\sin\varphi\sin\theta & 4\sin\varphi\cos\theta & 0 \end{vmatrix} =$$
   $$\left\langle 16\sin^2\varphi\cos\theta, 16\sin^2\varphi\sin\theta, 16\sin\varphi\cos\varphi \right\rangle.$$
   - So, we see $\left|\left|\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial\theta}\right|\right| = 16\sin\varphi$ (after simplifying!).
   - The integral is therefore $\displaystyle\int_0^{2\pi}\int_0^{\pi/2} 4\cos\varphi \cdot 16\sin\varphi\, d\varphi\, d\theta$.

Computing Surface Integrals, XII

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

3. Evaluate $\iint_S f \, d\sigma$.

- The integral is $\displaystyle\int_0^{2\pi} \int_0^{\pi/2} 4\cos\varphi \cdot 16\sin\varphi \, d\varphi \, d\theta$.

Computing Surface Integrals, XII

Example: Consider the function $f(x, y, z) = z$ on the upper half S of the sphere $x^2 + y^2 + z^2 = 16$.

3. Evaluate $\iint_S f \, d\sigma$.
   - The integral is $\displaystyle\int_0^{2\pi} \int_0^{\pi/2} 4\cos\varphi \cdot 16\sin\varphi \, d\varphi \, d\theta$.
   - Now we just evaluate it:

$$\int_0^{2\pi} \int_0^{\pi/2} 4\cos\varphi \cdot 16\sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} 32\sin^2\varphi \Big|_{\varphi=0}^{\pi/2} d\theta$$
$$= \int_0^{2\pi} 32 \, d\theta = 64\pi.$$

Computing Surface Integrals, XIII

We can also calculate surface integrals over implicit surfaces:

Proposition (Implicit Surface Integrals)

If $f(x,y,z)$ is continuous on the implicit surface S defined by $g(x,y,z) = c$, R is the projection of S into the xy-plane, and $\partial g/\partial z \neq 0$ on R, then the surface integral of f on S is

$$\iint_S f(x,y,z)\, d\sigma \;=\; \iint_R f(x,y,z)\, \frac{||\nabla g||}{|\nabla g \cdot \mathbf{k}|}\, dy\, dx$$

where ∇g is the gradient of g and $\mathbf{k} = \langle 0,0,1 \rangle$.
(Thus, $\nabla g \cdot \mathbf{k} = \partial g/\partial z$.)

The statement that $\partial g/\partial z \neq 0$ on R is equivalent to saying that the tangent plane to $g(x,y,z) = c$ is never vertical above R. In particular this implies that the surface never "doubles back" on itself over the region R.

Computing Surface Integrals, XIV

Example: Integrate the function $f(x, y, z) = 8xy$ over the portion of the plane $2x + y + 2z = 1$ with $0 \le x \le 1$, $0 \le y \le 1$.

Computing Surface Integrals, XIV

Example: Integrate the function $f(x, y, z) = 8xy$ over the portion of the plane $2x + y + 2z = 1$ with $0 \le x \le 1$, $0 \le y \le 1$.

- We use the implicit formula with $g(x, y, z) = 2x + y + 2z - 1$.
- We have $\nabla g = \langle 2, 1, 2 \rangle$ so $||\nabla g|| = \sqrt{2^2 + 1^2 + 2^2} = 3$ and $|\nabla g \cdot \mathbf{k}| = 2$.
- The desired integral is therefore $$\int_0^1 \int_0^1 8xy \cdot \frac{3}{2}\, dy\, dx = \int_0^1 6xy^2 \Big|_{y=0}^{1} dx = \int_0^1 6x\, dx = 3.$$

Computing Surface Integrals, XV

Example: Integrate the function $f(x, y, z) = xz$ over the portion of the plane $4x + 2y + z = 1$ where $0 \leq x \leq 1$, $0 \leq y \leq x$.

Computing Surface Integrals, XV

Example: Integrate the function $f(x, y, z) = xz$ over the portion of the plane $4x + 2y + z = 1$ where $0 \le x \le 1$, $0 \le y \le x$.

- We use the implicit formula with $g(x, y, z) = 4x + 2y + z - 1$.
- We have $\nabla g = \langle 4, 2, 1 \rangle$ so $||\nabla g|| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$ and $|\nabla g \cdot \mathbf{k}| = 1$.
- Since the function involves z, we must use the implicit relation to eliminate it. In this case, $z = 1 - 4x - 2y$, so $f(x, y, z) = xz = x - 4x^2 - 2xy$.
- The desired integral is therefore
$$\int_0^1 \int_0^x (x - 4x^2 - 2xy) \cdot \sqrt{21}\, dy\, dx$$
$$= \int_0^1 (xy - 4x^2 y - xy^2)\sqrt{21}\Big|_{y=0}^{x} dx$$
$$= \int_0^1 (x^2 - 5x^3)\sqrt{21}\, dx = -\frac{11}{12}\sqrt{21}.$$

Surface Areas and Average Values, I

We can use surface integrals to compute surface areas and average values.

- For surface area, the principle is the same as with finding areas using double integrals, finding volumes using triple integrals, and finding arclengths using line integrals: to find the area of a surface, we simply integrate the function 1 on the surface.
- For average value, the principle is also the same: to find the average value of f on a surface, we integrate f on the surface and then divide by the surface area.
- Depending on how the surface is described, it may be easier to set up the surface integral using a parametrization or easier to set it up using an implicit equation.

Surface Areas and Average Values, II

Example: Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies outside $x^2 + y^2 = 4$ and inside $x^2 + y^2 = 16$.

1. Find a parametrization for S.
2. Find the area of S.
3. Find the average value of $f(x, y, z) = z^2$ on S.

Surface Areas and Average Values, II

Example: Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies outside $x^2 + y^2 = 4$ and inside $x^2 + y^2 = 16$.

1. Find a parametrization for S.
2. Find the area of S.
3. Find the average value of $f(x, y, z) = z^2$ on S.

- We can use cylindrical coordinates to describe S.
- Explicitly, we can parametrize this portion of the cone as $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ for $2 \le r \le 4$ and $0 \le \theta \le 2\pi$.
- Then the area is given by $SA = \iint_S 1\, d\sigma$, and then the average value we want is given by $\dfrac{1}{\text{Area}(S)} \iint_S f\, d\sigma$.

Surface Areas and Average Values, III

Example: Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies outside $x^2 + y^2 = 4$ and inside $x^2 + y^2 = 16$.

2. Find the area of S.

- Note $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r \rangle$ with $2 \le r \le 4$, $0 \le \theta \le 2\pi$.

Surface Areas and Average Values, III

Example: Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies outside $x^2 + y^2 = 4$ and inside $x^2 + y^2 = 16$.

2. Find the area of S.

- Note $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ with $2 \le r \le 4$, $0 \le \theta \le 2\pi$.
- Then $\dfrac{\partial \mathbf{r}}{\partial r} = \langle \cos\theta, \sin\theta, 1\rangle$ and $\dfrac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$, so
$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle.$$
- So, we see $\left\| \dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta} \right\| = r\sqrt{2}$.
- The area is $\displaystyle\int_0^{2\pi} \int_2^4 r\sqrt{2}\, dr\, d\theta = \int_0^{2\pi} 6\sqrt{2}\, d\theta = 12\pi\sqrt{2}$.

Surface Areas and Average Values, IV

Example: Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies outside $x^2 + y^2 = 4$ and inside $x^2 + y^2 = 16$.

3. Find the average value of $f(x, y, z) = z^2$ on S.

- Note $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ with $2 \le r \le 4$, $0 \le \theta \le 2\pi$.
- We also calculated $\left|\left|\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta}\right|\right| = r\sqrt{2}$.

Surface Areas and Average Values, IV

Example: Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies outside $x^2 + y^2 = 4$ and inside $x^2 + y^2 = 16$.

3. Find the average value of $f(x, y, z) = z^2$ on S.

- Note $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ with $2 \le r \le 4$, $0 \le \theta \le 2\pi$.
- We also calculated $\left|\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right|\right| = r\sqrt{2}$.
- The function here is $z^2 = r^2$. So the average value is

$$\begin{aligned}
\frac{1}{12\pi\sqrt{2}} \int_0^{2\pi} \int_2^4 r^2 \cdot r\sqrt{2}\, dr\, d\theta &= \frac{1}{12\pi\sqrt{2}} \int_0^{2\pi} \frac{r^4}{4}\sqrt{2}\Big|_{r=2}^{4} d\theta \\
&= \frac{1}{12\pi\sqrt{2}} \int_0^{2\pi} 60\sqrt{2}\, d\theta \\
&= \frac{120\pi\sqrt{2}}{12\pi\sqrt{2}} = 10.
\end{aligned}$$

Mass and Center of Mass, I

Like with double, triple, and line integrals, we have mass and moment formulas for surface integrals:

Center of Mass and Moment Formulas (Thin Surface): Given a surface S of variable density $\delta(x, y, z)$ in 3-space:

- The total mass M is given by $M = \iint_S \delta(x, y, z)\, d\sigma$.
- The x-moment M_{yz} is given by $M_{yz} = \iint_S x\, \delta(x, y, z)\, d\sigma$.
- The y-moment M_{xz} is given by $M_{xz} = \iint_S y\, \delta(x, y, z)\, d\sigma$.
- The z-moment M_{xy} is given by $M_{xy} = \iint_S z\, \delta(x, y, z)\, d\sigma$.
- The center of mass $(\bar{x}, \bar{y}, \bar{z})$ has coordinates $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$.

Mass and Center of Mass, II

Example: A hill is shaped like the portion of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$, with all coordinates measured in meters. Snow accumulates on the hill such that the density is $\sqrt{17 - 4z}$ grams per square meter at height z. Find the total amount of snow on the hill.

Mass and Center of Mass, II

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- We are given the density of snow and want to compute the total mass, which is given by the integral $\iint_S \sqrt{17 - 4z}\, d\sigma$ where S is the surface representing the hill.
- By using cylindrical coordinates, we can parametrize the hill as $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), 4 - r^2 \rangle$ for $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Mass and Center of Mass, III

<u>Example</u>: The parametrization $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4 - r^2\rangle$m for $0 \le r \le 2$, $0 \le \theta \le 2\pi$ describes a hill. Snow accumulates on the hill such that the density is $\sqrt{17 - 4z}$ grams per square meter at height z. Find the total amount of snow on the hill.

Mass and Center of Mass, III

Example: The parametrization $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 4-r^2\rangle$m for $0 \le r \le 2$, $0 \le \theta \le 2\pi$ describes a hill. Snow accumulates on the hill such that the density is $\sqrt{17-4z}$ grams per square meter at height z. Find the total amount of snow on the hill.

- We have $\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle 2r^2\cos(\theta),\ 2r^2\sin(\theta),\ r\rangle$, so $\left|\left|\dfrac{\partial \mathbf{r}}{\partial r} \times \dfrac{\partial \mathbf{r}}{\partial \theta}\right|\right| = r\sqrt{4r^2+1}$.
- We also have $f(x,y,z) = \sqrt{17-4(4-r^2)} = \sqrt{4r^2+1}$.
- The integral is $\int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\cdot r\sqrt{4r^2+1}\,dr\,d\theta$ $= \int_0^{2\pi}\int_0^2 (r+4r^3)\,dr\,d\theta = \int_0^{2\pi} 18\,d\theta = 36\pi$.
- This means there are 36π g of snow on the hill.

Summary

We discussed how to set up and evaluate surface integrals on parametric surfaces and on implicit surfaces.

We discussed some applications of surface integrals to computing surface area, average value, mass, and center of mass.

Next lecture: Vector fields, work, circulation, and flux.