Math 2321 (Multivariable Calculus)

Lecture #26 of 37 ~ March 22, 2021

Parametric Surfaces + Surface Integrals

- Parametric Surfaces
- Finding Parametrizations of Surfaces

This material represents §4.2.1 from the course notes.

Exam Logistics

The exam is graded and the grades were posted over the weekend.

- Solutions are also posted on the course webpage.
- If you have any questions about the exam grading, etc., please let me know during office hours / after class.

Now for the good news: there is no class on Wednesday, because it is a campus holiday.

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Now for the good news: there is no class on Wednesday, because it is a campus holiday.

Now for the bad news: because this holiday was announced less than two weeks ago, this has screwed up the schedule. As such, we won't cover the material for WeBWorK 9 until Thursday this week. Therefore, I am moving the due dates of WeBWorKs 9 and 10 to Sunday at 5am and WeBWorK 11 to Friday at 5am.

Surfaces, I

We would now like to consider the problem of computing the integral of a function on a surface in 3-dimensional space.

- One motivating application is to calculate the surface area of a given surface.
- Another application is to find the average value of a function on a surface, or to compute the mass of a surface region with variable density.
- In a similar way to how we computed line integrals using (single) integrals, we will be able to compute surface integrals as double integrals.

Surfaces, II

We have essentially two ways to describe a surface in 3-space algebraically:

1. As an implicit surface of the form $f(x, y, z) = c$ for some function $f(x, y, z)$ and some constant c.
2. As a parametric surface $\mathbf{r}(s, t) = \langle x(s, t),\ y(s, t),\ z(s, t)\rangle$ for two parameters s and t. As s and t vary, we can graph the vector-valued function $\mathbf{r}$ to obtain a surface in space.

- Note that the "explicit surface" $z = g(x, y)$ is simply a special case of the general implicit surface, since $g(x, y) - z = 0$ has the form $f(x, y, z) = c$ with $f(x, y, z) = g(x, y) - z$ and $c = 0$.
- We have already talked a bit about implicit surfaces: for example, we described how to find the tangent plane to an implicit surface using the gradient back in chapter 2.

Parametric Surfaces, I

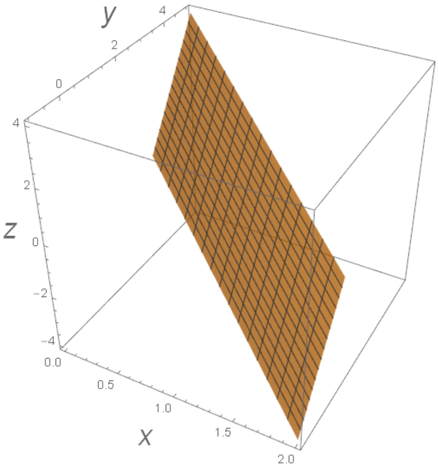
We will now spend some time discussing parametrizations of surfaces.

- Parametric descriptions of surfaces are often easier to work with than implicit descriptions.
- For example, graphing a parametric surface $\mathbf{r}(s,t) = \langle x(s,t),\, y(s,t),\, z(s,t)\rangle$ requires only plugging in values for (s,t) and plotting the resulting points (x,y,z).
- In contrast, graphing an implicit surface requires finding solutions to an implicit equation, which is typically much harder.

Let's (have a computer) draw some parametric surfaces.

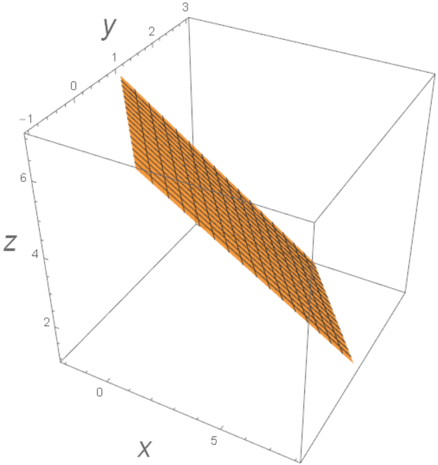
Parametric Surfaces, II: It's A Bird! It's A Plane!

Example: $\mathbf{r}(s,t) = \langle 1-s, 2+2s-t, 2s-2t \rangle$ is a plane:



Parametric Surfaces, III: It's A Bird! It's Also A Plane!

<u>Example</u>: $\mathbf{r}(s,t) = \langle 3 - 2s + 3t, 1 + s + t, 4 - 3t \rangle$ is also a plane:



Parametric Surfaces, IV

More generally, the parametric surface
$\mathbf{r}(s,t) = \langle x_0, y_0, z_0 \rangle + t\,\langle v_1, v_2, v_3 \rangle + s\,\langle w_1, w_2, w_3 \rangle$ will be a plane.

- Specifically, it will be the plane that passes through the point (x_0, y_0, z_0) and contains the two vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$.
- Here, we need $\mathbf{v}$ and $\mathbf{w}$ not to be parallel, otherwise the graph will degenerate to a line.

Any particular plane will have many different parametrizations.

- For example, the two parametrizations
  $\mathbf{r}(s,t) = \langle s,\ t,\ 1 - s - t \rangle$ and
  $\mathbf{r}(s,t) = \langle -3 + s - 2t,\ 2 + t + 2s,\ 2 + t - 3s \rangle$ actually describe the same plane $x + y + z = 1$.

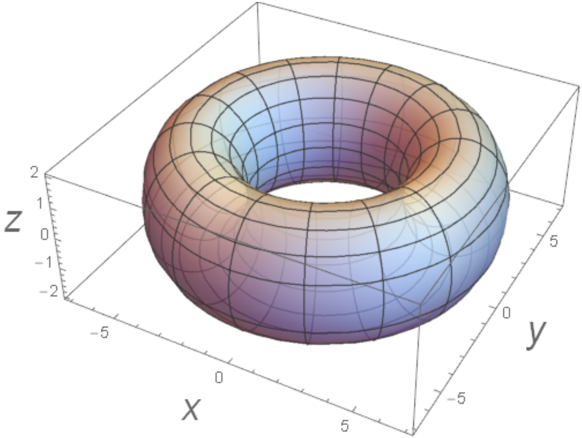
Parametric Surfaces, V

As we discussed back in chapter 1, we can also describe a plane as an implicit surface.

- Indeed, we could also describe the plane $\mathbf{r}(s,t) = \langle x_0, y_0, z_0 \rangle + t\mathbf{v} + s\mathbf{w}$ using the equation $ax + by + cz = d$, where $\langle a, b, c \rangle = \mathbf{v} \times \mathbf{w}$ is the normal vector to the plane and $d = ax_0 + by_0 + cz_0$.
- For example, our first plane $\mathbf{r}(s,t) = \langle 1 - s, 2 + 2s - t, 2s - 2t \rangle$ has equation $-2x - 2y + z = -6$, as we can see by computing the cross product $\langle -1, 2, 2 \rangle \times \langle 0, -1, -2 \rangle = \langle -2, -2, 1 \rangle$.

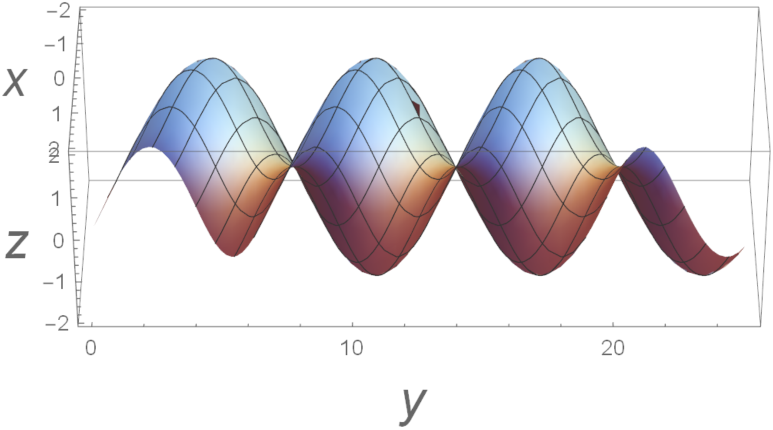
Parametric Surfaces, VI: Mmm... Donuts....

Example: For $0 \le t \le 2\pi$ and $0 \le s \le 2\pi$, the surface $\mathbf{r}(s,t) = \langle \cos(t)(5+3\sin(s)), \sin(t)(5+3\cos(s)), 3\sin(s) \rangle$ is a torus:



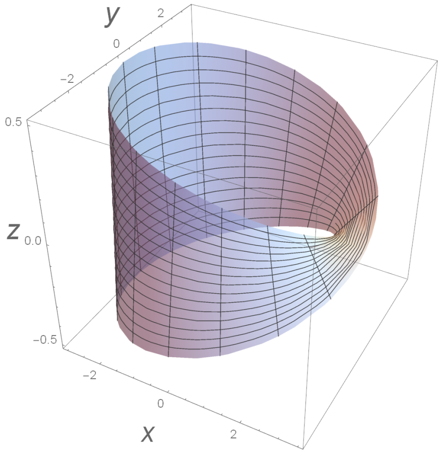
Parametric Surfaces, VII

Example: For $0 \leq t \leq 4\pi$ and $0 \leq s \leq 4\pi$,
$\mathbf{r}(s,t) = \langle \cos(s) + \cos(t),\ s+t,\ \sin(s) + \sin(t) \rangle$ is a helical ribbon:



Parametric Surfaces, VIII: On The One Side

Example: For $0 \leq t \leq \pi$ and $-1/2 \leq r \leq 1/2$, $\mathbf{r}(s,t) = \langle (3 + r\cos t)\cos(2t), (3 + r\cos t)\sin(2t), r\sin t\rangle$ is a Möbius strip:



Parametric Surfaces, IX

In general, it can be a somewhat involved problem to convert a geometric or verbal description of a surface into a parametrization: it is really more of an art form[1] than a general procedure.

- To parametrize parts of cylinders, cones, and spheres, it is almost always a very good idea to consider whether cylindrical or spherical coordinates can be of assistance.
- Using translations and rescalings, we can also parametrize surfaces like ellipsoids.

There are many different ways to parametrize the same surface, and which description is best will depend on what the parametrization will be used for. (We will illustrate with examples.)

[1] Indeed, in the hands of people who do digital 3D graphics, it can quite literally be an art form!

Parametric Surfaces, X

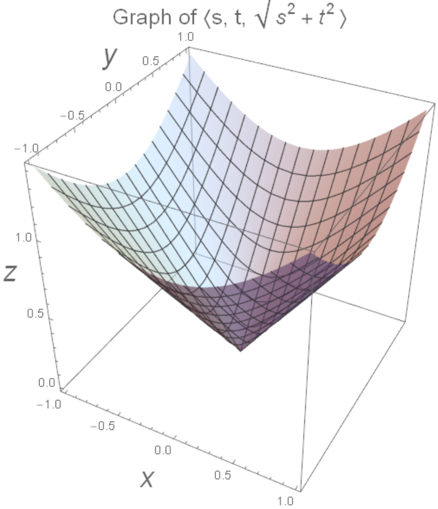
Example: Parametrize the portion of the cone $z = \sqrt{x^2 + y^2}$ lying above the plane region with $-1 \le x \le 1$, $-1 \le y \le 1$.

- Here, we can just take our parameters to be the variables x and y, because we want the surface on a range where x and y are both bounded by constants.
- So, if we take $x = s$ and $y = t$, we get the parametrization $\mathbf{r}(s,t) = \langle s, t, \sqrt{s^2 + t^2} \rangle$ for $-1 \le s \le 1$, $-1 \le t \le 1$.

By using a computer, we can use the parametrization we've found to draw the graph of the part of the surface we want.

Parametric Surfaces, XI

Here is $\mathbf{r}(s,t) = \langle s, t, \sqrt{s^2+t^2} \rangle$ for $-1 \le s \le 1$, $-1 \le t \le 1$:



Parametric Surfaces, XII

Example: Parametrize the portion of the cone $z = \sqrt{x^2 + y^2}$ where $z \le \sqrt{2}$.

- If we try to use $x = s$ and $y = t$ like we did before, then the portion of the surface we want is the part where $\sqrt{s^2 + t^2} \le \sqrt{2}$, which is to say, where $s^2 + t^2 \le 2$.
- We can certainly describe this region by putting bounds on s and t: for example, we could take $-\sqrt{2} \le s \le \sqrt{2}$ and then $-\sqrt{2 - s^2} \le t \le \sqrt{2 - s^2}$.
- But this is rather ugly and is not likely to be pleasant if we have to use it for something.

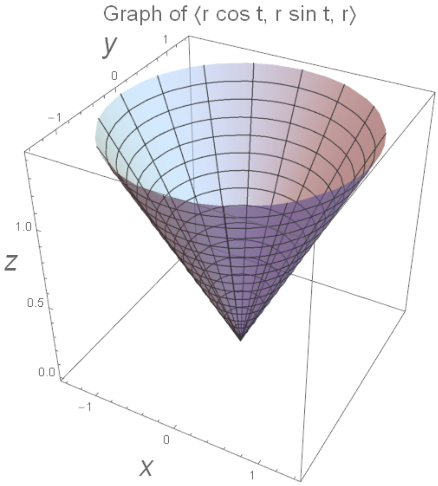
Parametric Surfaces, XIII

Example: Parametrize the portion of the cone $z = \sqrt{x^2 + y^2}$ where $z \leq \sqrt{2}$.

- Instead, we can take a cue from cylindrical coordinates: in cylindrical, this cone has the very nice equation $z = r$.
- So since z is determined, let's try using r and θ as our parameters.
- Because $x = r\cos\theta$ and $y = r\sin\theta$, we know what x and y are, and also from the surface equation, we need $z = r$.
- Furthermore, the part with $z \leq \sqrt{2}$ corresponds to $0 \leq r \leq \sqrt{2}$. There's no restriction on θ so we have $0 \leq \theta \leq 2\pi$.
- So our parametrization is $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ for $0 \leq r \leq \sqrt{2}$, $0 \leq \theta \leq 2\pi$.

Parametric Surfaces, XIV

Here is $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ for $0 \le r \le \sqrt{2}$, $0 \le \theta \le 2\pi$:



Parametric Surfaces, XV

Example: Parametrize the portion of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = -2$ and $z = 2$.

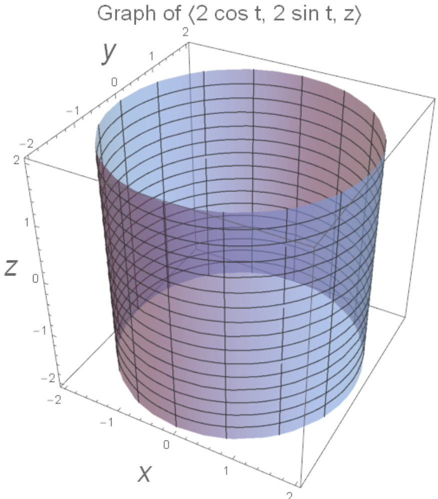
Parametric Surfaces, XV

Example: Parametrize the portion of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = -2$ and $z = 2$.

- In cylindrical, the cylinder is $r = 2$. That leaves the two parameters θ and z, so we write everything in terms of those.
- Since $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, a parametrization of the full cylinder is $x = 2\cos\theta$, $y = 2\sin\theta$, $z = z$, where $0 \le \theta \le 2\pi$.
- Then to get the desired portion, we just take $-2 \le z \le 2$.
- So our parametrization is $\mathbf{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$ where $0 \le \theta \le 2\pi$ and $-2 \le z \le 2$.

Parametric Surfaces, XVI

Here's a plot of $\mathbf{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$ where $0 \le \theta \le 2\pi$ and $-2 \le z \le 2$:



Parametric Surfaces, XVII

We will note that we can call the parameters anything we want.

- Often, when the parametrization is motivated by cylindrical or spherical coordinates, we will just use those variable names.
- However, when some of the coordinates are the same as the ones we are using, it can be a little bit confusing to write things like "$z = z$".
- For this reason we will often call our parameters s and t, so as to avoid this kind of re-use of variable labels.
- So, we could also parametrize the cylinder $x^2 + y^2 = 4$ as $x = 2\cos t$, $y = 2\sin t$, $z = s$, where $0 \le t \le 2\pi$ and $-2 \le s \le 2$.

Parametric Surfaces, XVIII

Example: Parametrize the portion of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = y - 2$ and $z = x + 4$.

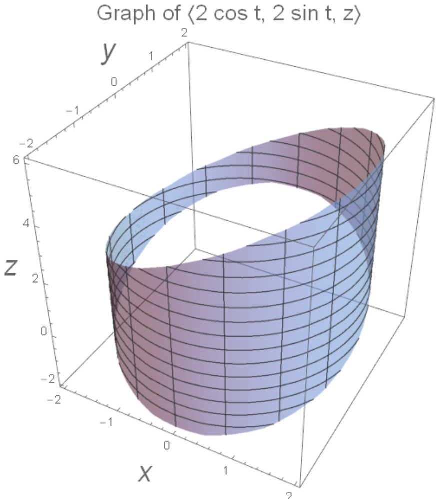
Parametric Surfaces, XVIII

Example: Parametrize the portion of the cylinder $x^2 + y^2 = 4$ lying between the planes $z = y - 2$ and $z = x + 4$.

- Like in the previous example, we take the parametrization of the full cylinder as $\mathbf{r}(s, t) = \langle 2\cos t, 2\sin t, s \rangle$, and then restrict the ranges for s and t appropriately.
- In this case, we want the portion of the surface where $y - 2 \leq z \leq x + 4$.
- It is straightforward to check that the two planes do not intersect inside the cylinder (since $y - 2 \leq 0$ inside the cylinder, while $x + 4 \geq 2$).
- So in this case, we take $0 \leq t \leq 2\pi$ and $2\sin t - 2 \leq s \leq 2\cos t + 4$.

Parametric Surfaces, XIX

Here's a plot of $\mathbf{r}(s,t) = \langle 2\cos t, 2\sin t, s\rangle$ where $0 \le t \le 2\pi$ and $2\sin t \le s \le 2\cos t + 4$:



Parametric Surfaces, XX

Example: Parametrize the sphere $x^2 + y^2 + z^2 = 9$.

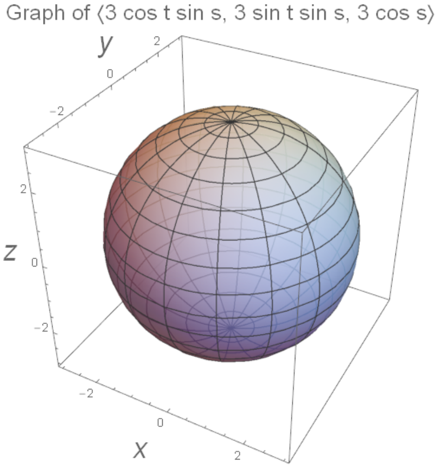
Parametric Surfaces, XX

Example: Parametrize the sphere $x^2 + y^2 + z^2 = 9$.

- We could use rectangular coordinates. But this would require splitting the sphere into halves.
- Instead, we use spherical coordinates, which is far more convenient. In spherical, $x = \rho \cos(\theta) \sin(\varphi)$, $y = \rho \sin(\theta) \sin(\varphi)$, $z = \rho \cos(\varphi)$.
- The sphere has equation $\rho = 3$.
- So, if we think of t as θ and s as φ, we get $\mathbf{r}(s,t) = \langle 3\cos(t)\sin(s), 3\sin(t)\sin(s), 3\cos(s)\rangle$, with $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

Parametric Surfaces, XXI

Here's a plot of $\mathbf{r}(s,t) = \langle 3\cos(t)\sin(s), 3\sin(t)\sin(s), 3\cos(s)\rangle$ with $0 \le t \le 2\pi$ and $0 \le s \le \pi$:



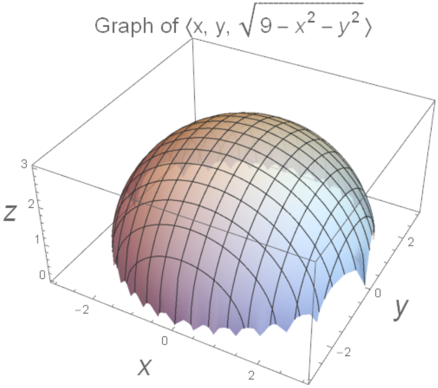
Parametric Surfaces, XXII

Now, we could have described the sphere using the rectangular equation $x^2 + y^2 + z^2 = 9$.

- If we want the top half of the sphere, we can describe it as $\mathbf{r}(s,t) = \langle x, y, \sqrt{9 - x^2 - y^2} \rangle$, where $-3 \leq x \leq 3$, $-\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}$.
- However, a peculiar thing will happen if you ask a computer to draw the surface this way. (See if you can guess.)

Parametric Surfaces, XXIII

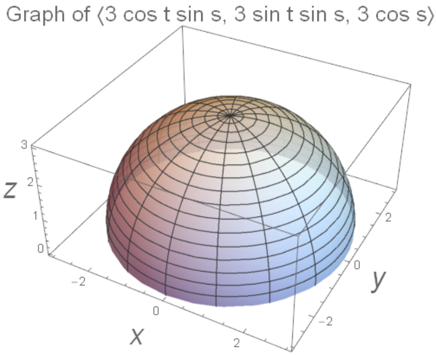
Here's a plot of $\mathbf{r}(s,t) = x, y, \sqrt{9 - x^2 - y^2}\rangle$ where $-3 \le x \le 3$ and $-\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}$:



Notice the jagged nature of the plot near $z = 0$. This is because of how computers plot the graph by plugging in points.

Parametric Surfaces, XXIV

Compare to the plot using the spherical-coordinate parametrization $\mathbf{r}(s,t) = \langle 3\cos(t)\sin(s), 3\sin(t)\sin(s), 3\cos(s)\rangle$ but with $0 \le t \le 2\pi$ and $0 \le s \le \pi/2$ for just the top half:



Much nicer looking, isn't it? (Also, compare the mesh lines.)

Parametric Surfaces, XXV

Example: Parametrize the portion of the saddle surface $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 1$.

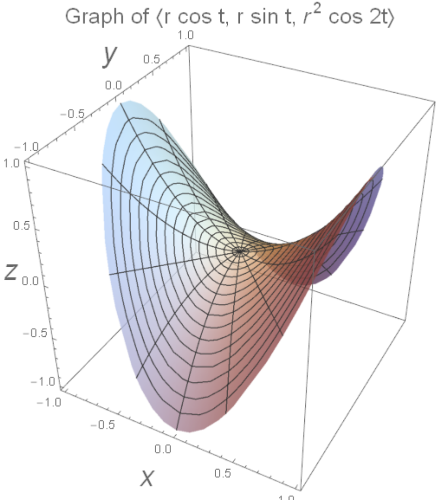
Parametric Surfaces, XXV

Example: Parametrize the portion of the saddle surface $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 1$.

- We can use cylindrical coordinates here, because the condition corresponds to $r \le 1$.
- In cylindrical, the surface is $z = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$. So we take our parameters as r and θ.
- Our parametrization is then $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\cos 2\theta\rangle$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

Parametric Surfaces, XXVI: Saddle Up

Here's a plot of $\mathbf{r}(r, \theta) = \langle r \cos\theta, r \sin\theta, r^2 \cos\theta \sin\theta \rangle$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$:



Parametric Surfaces, XXVII

We can also use translations and scalings of some of our "basic parametrizations" to describe other surfaces.

- By adding or subtracting constants from the coordinates, we can "recenter" the surface anywhere we like.
- The idea is similar to how we can parametrize the circle $(x-h)^2 + (y-k)^2 = r^2$ as $x = h + r\cos t$, $y = k + r\sin t$: by shifting the coordinates, we can describe circles with different centers.
- We can also rescale coordinates to stretch or compress the surface along a coordinate direction.
- The idea is similar to the way we can parametrize the ellipse $x^2/a^2 + y^2/b^2 = 1$ as $x = a\cos t$, $y = b\sin t$: by rescaling the coordinates, we can stretch the circle in the x- or y-direction to make an ellipse.

Parametric Surfaces, XXVIII

Example: Parametrize the sphere $(x-2)^2+(y+1)^2+(z-6)^2=4$.

Parametric Surfaces, XXVIII

Example: Parametrize the sphere $(x-2)^2+(y+1)^2+(z-6)^2=4$.

- It is not so easy to describe this sphere using spherical coordinates directly.
- However, if we shift the coordinates by setting $x' = x - 2$, $y' = y + 1$, and $z' = z - 6$, then we see $(x')^2 + (y')^2 + (z')^2 = 4$, and we can use spherical coordinates to parametrize x', y', z'.
- The results for x, y, z are
  $x = 2 + 2\cos(t)\sin(s)$,
  $y = -1 + 2\sin(t)\sin(s)$,
  $z = 6 + 2\cos(s)$,
  with $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

Parametric Surfaces, XXIX

Example: Parametrize the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

Parametric Surfaces, XXIX

<u>Example</u>: Parametrize the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

- It is again not so easy to write down the parametrization using any of our coordinate systems directly.
- However, if we rescale the coordinates by setting $x' = x/2$, $y' = y/3$, and $z' = z/4$, then we see $(x')^2 + (y')^2 + (z')^2 = 1$, and we can use spherical coordinates to parametrize x', y', z'.
- The results for x, y, z are
  $x = 2\cos(t)\sin(s)$,
  $y = 3\sin(t)\sin(s)$,
  $z = 4\cos(s)$,
  with $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

Parametric Surfaces, XXX

If we have a parametrization of a surface, we can use it to find the tangent plane to the surface at a given point.

- The key observation is that if the surface S is parametrized by the vector-valued function $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$, then the two partial derivatives $\mathbf{r}_s = \dfrac{\partial \mathbf{r}}{\partial s}$ and $\mathbf{r}_t = \dfrac{\partial \mathbf{r}}{\partial t}$ are both tangent to the surface.
- Therefore, the cross product $\dfrac{\partial \mathbf{r}}{\partial s} \times \dfrac{\partial \mathbf{r}}{\partial t}$ will be perpendicular to the tangent plane, and is thus a normal vector for the tangent plane.

Parametric Surfaces, XXXI

Example: Find an equation for the tangent plane to the surface $\mathbf{r}(s,t) = \langle s\cos(t), s\sin(t), s^2 \rangle$ when $s = 1$ and $t = \pi/2$.

Parametric Surfaces, XXXI

Example: Find an equation for the tangent plane to the surface $\mathbf{r}(s,t) = \left\langle s\cos(t), s\sin(t), s^2 \right\rangle$ when $s = 1$ and $t = \pi/2$.

- We compute $\mathbf{r}_s(s,t) = \langle \cos t, \sin t, 2s \rangle$ and $\mathbf{r}_t(s,t) = \langle -s\sin t, s\cos t, 0 \rangle$.
- Then $\mathbf{r}_s(1,\pi/2) = \langle 0,1,2 \rangle$, and $\mathbf{r}_t(1,\pi/2) = \langle -1,0,0 \rangle$.
- So, the normal vector to the tangent plane is $\mathbf{n} = \langle 0,1,2 \rangle \times \langle -1,0,0 \rangle = \langle 0,-2,1 \rangle$.
- The tangent plane passes through the point on the surface where $s = 1$ and $t = \pi/2$, which is $\mathbf{r}(1,\pi/2) = \langle 0,1,1 \rangle$.
- Thus, an equation for the tangent plane is given by $0(x-0) - 2(y-1) + 1(z-1) = 0$ or equivalently $2y - z = 1$.

Parametric Surfaces, XXXII

Example: Find an equation for the plane tangent to the surface $\mathbf{r}(s, t) = \langle s^2, 2st, t^3 \rangle$ at the point $(4, 4, -1)$.

Parametric Surfaces, XXXII

Example: Find an equation for the plane tangent to the surface $\mathbf{r}(s,t) = \langle s^2, 2st, t^3 \rangle$ at the point $(4, 4, -1)$.

- First, we need to find the values of s and t at $(4, 4, -1)$.
- If $\langle 4, 4, -1 \rangle = \langle s^2, 2st, t^3 \rangle$ then we see $t^3 = -1$ so $t = -1$, and then $2st = 4$ gives $s = -2$.
- Now, we have $\mathbf{r}_s(s,t) = \langle 2s, 2t, 0 \rangle$ and $\mathbf{r}_t(s,t) = \langle 0, 2s, 3t^2 \rangle$, so $\mathbf{r}_s(-2,-1) = \langle -4, -2, 0 \rangle$ and $\mathbf{r}_t(-2,-1) = \langle 0, -4, 3 \rangle$.
- Thus, the normal vector to the tangent plane is $\mathbf{n} = \langle -4, -2, 0 \rangle \times \langle 0, -4, 3 \rangle = \langle -6, 12, 16 \rangle$.
- Thus, an equation for the tangent plane is given by $-6(x-4) + 12(y-4) + 16(z+1) = 0$ or equivalently $-6x + 12y + 16z = 8$.

Summary

We discussed parametric surfaces and some of their properties.

We discussed how to parametrize a range of different surfaces.

We discussed how to find the tangent plane to a parametric surface.

Next lecture: Surface integrals.

Note that there are no classes on Wednesday, so next lecture is on Thursday. The WeBWorK is also extended to Sunday at 5am. Office hours will not be held on Wednesday but will run Thursday as normal. So, please, enjoy your Wednesday off!