Math 2321 (Multivariable Calculus)

Lecture #25 of 37 $\sim$ March 18, 2021

Midterm #2 Review #2

Midterm 2 Exam Topics

The topics for the exam are as follows:

- Lagrange multipliers
- Double integrals in rectangular coordinates
- Changing the order of integration
- Double integrals in polar coordinates
- Triple integrals in rectangular coordinates
- Triple integrals in cylindrical coordinates
- Triple integrals in spherical coordinates
- Areas, volumes, average value, mass, center of mass

This represents §2.6 + §3.1−3.2 + §3.3.2−§3.4 from the notes and WeBWorKs 5-8. Note that general changes of coordinates (§3.3.1) are NOT included.

Exam Information

The exam format is essentially the same as the first midterm.

- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- There are approximately 6 pages of material, about 1/5 multiple choice and the rest free response.
- The "official" exam time limit is 65+25 = 90 minutes, plus 30 minutes of turnaround time (not to be used for working).
- Collaboration of any kind is not allowed. You may not discuss anything about the exam with anyone other than me (the instructor) until 5pm Eastern on Tuesday, March 23rd. This includes Piazza posts.

I have sent Canvas notifications to everyone about their midterm 2 window. Please check to confirm it is the time you want.

Review Problems, I

(#5a) Reverse the order of integration for $\int_0^3 \int_0^{x^2} xy \, dy \, dx$.

Review Problems, I

(#5a) Reverse the order of integration for $\int_0^3 \int_0^{x^2} xy\,dy\,dx$.

- The region is defined by the inequalities $0 \le x \le 3$, $0 \le y \le x^2$.
- These curves intersect at $(0, 0)$ and $(3, 9)$.
- Therefore, when we reverse the order, y ranges from 0 to 9, while x ranges from $\sqrt{y}$ to 3.
- Hence the new integral is $\int_0^9 \int_{\sqrt{y}}^3 xy\,dx\,dy$.

Review Problems, II

(#2) You have 60 meters of fencing and wish to make a rectangular enclosure along a straight river, meaning that you only need to fence the east, west, and north sides (not the south side). What dimensions maximize the total area of the enclosure?

Review Problems, II

(#2) You have 60 meters of fencing and wish to make a rectangular enclosure along a straight river, meaning that you only need to fence the east, west, and north sides (not the south side). What dimensions maximize the total area of the enclosure?

- Suppose the length (east-west) is l meters and the width (north-south) is w meters.
- Then the area is $A = lw$ square meters and the total fence used is $2l + w$ meters.
- So we want to maximize $A = lw$ subject to $2l + w = 60\,\text{m}$.
- Using Lagrange multipliers yields the system gives $w = 2\lambda$, $l = \lambda$, $2l + w = 60$.
- Plugging the first two equations into the third yields $4\lambda = 60$, so $\lambda = 15$.
- Then $l = 15\,\text{m}$, $w = 30\,\text{m}$. This is easily seen to be the maximum by the physical setup of the problem.

Review Problems, III

(#10a) Compute $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{x+y} \sqrt{x^2+y^2}\, dz\, dy\, dx$ by converting to cylindrical or spherical.

Review Problems, III

(#10a) Compute $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{x+y} \sqrt{x^2+y^2}\, dz\, dy\, dx$ by converting to cylindrical or spherical.

- Here, we want to use cylindrical, since the function and z-limits will be nice in cylindrical.
- The limits in x and y describe the interior of the circle $x^2 + y^2 = 1$, which is $0 \le \theta \le 2\pi$, $0 \le r \le 1$ in polar.
- The z limits are then $x + y = r\cos\theta + r\sin theta$ and $x^2 + y^2 = r^2$. The differential is $dV = r\, dz\, dr\, d\theta$.
- So the integral is $\int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{r\cos\theta + r\sin\theta} r \cdot r\, dz\, dr\, d\theta =$ $\int_{0}^{2\pi} \int_{0}^{1} r^3(\sin\theta + \cos\theta)\, dr\, d\theta = \int_{0}^{2\pi} \frac{1}{4}(\sin\theta + \cos\theta)\, d\theta = 0.$

Review Problems, IV

(#6c) Set up and evaluate a double integral in polar coordinates for the volume under $z = 4 - x^2 - y^2$ and above the xy-plane.

Review Problems, IV

(#6c) Set up and evaluate a double integral in polar coordinates for the volume under $z = 4 - x^2 - y^2$ and above the xy-plane.

- A quick sketch shows that the region underneath the surface is the interior of the circle $x^2 + y^2 = 4$.
- So, in polar, this is the region with $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$.
- Then the function, representing the height, is $f(x, y) = 4 - x^2 - y^2 = 4 - r^2$.
- The differential is $dA = r\, dr\, d\theta$.
- Thus, the integral is $\int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta$.
- Evaluating yields $\int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} 12\, d\theta = 24\pi$.

Review Problems, V

(#10c) Evaluate $\int_{0}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-1}^{x^2+y^2} \frac{1}{\sqrt{x^2+y^2}} \, dz \, dy \, dx$ by converting to cylindrical or spherical.

Review Problems, V

(#10c) Evaluate $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-1}^{x^2+y^2} \frac{1}{\sqrt{x^2+y^2}} dz\, dy\, dx$ by converting to cylindrical or spherical.

- Here, we want to use cylindrical, since the function and z-limits both will be nice in cylindrical.
- The limits in x and y describe the portion inside $x^2 + y^2 = 9$ where $x \geq 0$, which is $-\pi/2 \leq \theta \leq pi/2$, $0 \leq r \leq 3$ in polar.
- The z limits are then -1 and $x^2 + y^2 = r^2$. The differential is $dV = r\, dz\, dr\, d\theta$.
- So the integral is $\int_{-\pi/2}^{\pi/2} \int_0^3 \int_{-1}^{r^2} \frac{1}{r} \cdot r\, dz\, dr\, d\theta =$

$$\int_{-\pi/2}^{\pi/2} \int_0^3 (r^2 + 1)\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} 12\, d\theta = 12\pi.$$

Review Problems, VI

(#1d) Find the minimum and maximum values of $f(x, y, z) = 2x + 4y + 5z$ subject to $x^2 + y^2 + z^2 = 1$.

Review Problems, VI

(#1d) Find the minimum and maximum values of $f(x, y, z) = 2x + 4y + 5z$ subject to $x^2 + y^2 + z^2 = 1$.

- We use Lagrange multipliers: here $f = 2x + 4y + 5z$ and $g = x^2 + y^2 + z^2$.
- The system is $\nabla f = \lambda \nabla g$ and $g = c$.
- System is $2 = 2\lambda x$, $4 = 2\lambda y$, $5 = 2\lambda z$, $x^2 + y^2 + z^2 = 1$.
- Thus $x = 2/(2\lambda)$, $y = 4/(2\lambda)$, $z = 5/(2\lambda)$, so $45/(4\lambda^2) = 1$.
- Then the last equation is $4/(4\lambda^2) + 16/(4\lambda^2) + 25/(4\lambda^2) = 1$, which yields $45/(4\lambda^2) = 1$.
- Thus, $\lambda = \pm\sqrt{45/4}$, yielding $(x, y, z) = \pm\frac{1}{\sqrt{45}}(2, 4, 5)$.
- Minimum is $-\sqrt{45}$ at $-\frac{1}{\sqrt{45}}(2, 4, 5)$.
  Maximum is $\sqrt{45}$ at $\frac{1}{\sqrt{45}}(2, 4, 5)$.

Review Problems, VII

(#6a) Set up the integral of $f(x, y) = x$ on the region inside $x^2 + y^2 = 1$ with $x \leq 0$ and $y \leq 0$, in polar coordinates.

Review Problems, VII

(#6a) Set up the integral of $f(x, y) = x$ on the region inside $x^2 + y^2 = 1$ with $x \leq 0$ and $y \leq 0$, in polar coordinates.

- The region is the interior of the unit circle in the third quadrant.
- In polar, this is described by $\pi \leq \theta \leq 3\pi/2$ and $0 \leq r \leq 1$.
- The function is $x = r\cos\theta$, and the differential is $dA = r\,dr\,d\theta$.
- Thus, the integral is $\displaystyle\int_{\pi}^{3\pi/2}\int_0^1 (r\cos\theta)\,r\,dr\,d\theta$.
- Evaluating yields
$$\int_{\pi}^{3\pi/2}\int_0^1 (r\cos\theta)\,r\,dr\,d\theta = \int_{\pi}^{3\pi/2}\frac{1}{3}\cos\theta\,d\theta = -\frac{1}{3}.$$

Review Problems, VIII

(#10d) Find $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} \frac{z^2}{\sqrt{x^2+y^2+z^2}} \, dz \, dy \, dx$ by converting to cylindrical or spherical.

Review Problems, VIII

(#10d) Find $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} \frac{z^2}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx$ by converting to cylindrical or spherical.

- Here, we want to use spherical, since the function and upper z-limit both involve spheres and ρ, and the lower z-limit is a cone.
- The region is the "ice cream cone" inside the sphere $\rho = 2$ and above the cone $\varphi = \pi/4$.
- The function is $\frac{z^2}{\sqrt{x^2+y^2+z^2}} = \frac{\rho^2\cos^2\varphi}{\rho}$.
- The differential is $dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$.
- So the integral is $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \frac{\rho^2\cos^2\varphi}{\rho}\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$

$$\int_0^{2\pi}\int_0^{\pi/4} 4\cos^2\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta = 4\pi(4-\sqrt{2})/3.$$

Review Problems, IX

(#11i) Set up the integral of x on the region with $x \geq 0$, $y \geq 0$, $z \geq 0$ and below $z = 4 - x - y^2$.

Review Problems, IX

(#11i) Set up the integral of x on the region with $x \geq 0$, $y \geq 0$, $z \geq 0$ and below $z = 4 - x - y^2$.

- We can use any coordinate system, but here rectangular is easiest (don't be fooled by the function: it's not $z = 4 - r^2$!).
- Drawing a quick sketch of the surface shows that x ranges from 0 to 4, and then in the xy-plane, y ranges from 0 to $\sqrt{4-x}$, and then z ranges from 0 to $4 - x - y^2$.
- Thus, the integral is $\int_0^4 \int_0^{\sqrt{4-x}} \int_0^{4-x-y^2} x \, dz \, dy \, dx$.

Review Problems, X

(#12b) Find the total mass and the center of mass for the solid bounded by $0 \le z \le \sqrt{x^2 + y^2} \le 2$ with density $d(x, y, z) = 2\sqrt{x^2 + y^2}\,\text{g/cm}^3$.

Review Problems, X

(#12b) Find the total mass and the center of mass for the solid bounded by $0 \leq z \leq \sqrt{x^2+y^2} \leq 2$ with density $d(x,y,z) = 2\sqrt{x^2+y^2}\,\text{g/cm}^3$.

- We want to use cylindrical coordinates. The region is $0 \leq z \leq r \leq 2$ with density $d = 2r$.
- Mass is $M = \iiint_D d(x,y,z)\,dV = \int_0^{2\pi} \int_0^2 \int_0^r 1 \cdot 2r\,dz\,dr\,d\theta = \frac{128\pi}{5}$ g.
- The moments for the center of mass are $M_x = \iiint_D x\,d(x,y,z)\,dV = \int_0^{2\pi} \int_0^2 \int_0^r r\cos\theta \cdot 2r\,dz\,dr\,d\theta = 0$, $M_y = \iiint_D y\,d(x,y,z)\,dV = \int_0^{2\pi} \int_0^2 \int_0^r r\sin\theta \cdot 2r\,dz\,dr\,d\theta = 0$, $M_z = \iiint_D z\,d(x,y,z)\,dV = \int_0^{2\pi} \int_0^2 \int_0^r z \cdot 2r\,dz\,dr\,d\theta = 64\pi/3$.
- So the center of mass is $\frac{1}{M}(M_x, M_y, M_z) = (0, 0, \frac{5}{6}\,\text{cm})$.

Review Problems, XI

(#6b) In polar coordinates, set up and then evaluate the integral $\iint_R \sqrt{x^2 + y^2}\, dA$ where R is the region inside $x^2 + y^2 = 16$, above $y = x$ and $y = -x$.

Review Problems, XI

(#6b) In polar coordinates, set up and then evaluate the integral $\iint_R \sqrt{x^2+y^2}\,dA$ where R is the region inside $x^2+y^2=16$, above $y=x$ and $y=-x$.

- The region is the interior of a quarter-circle, ranging from $\theta=\pi/4$ to $\theta=3\pi/4$.
- The function is $\sqrt{x^2+y^2}=r$, and the differential is $dA = r\,dr\,d\theta$.
- Therefore, the integral is $\int_{\pi/4}^{3\pi/4}\int_0^4 (r)\,r\,dr\,d\theta$.
- Evaluating yields $\int_{\pi/4}^{3\pi/4}\int_0^4 r^2\,dr\,d\theta = \int_{\pi/4}^{3\pi/4}\frac{64}{3}\,d\theta = \frac{32\pi}{3}$.

Review Problems, XII

(#1e) Find the minimum and maximum values of $f(x, y, z) = xyz$ subject to $x^2 + 4y^2 + 16z^2 = 48$.

Review Problems, XII

(#1e) Find the minimum and maximum values of $f(x, y, z) = xyz$ subject to $x^2 + 4y^2 + 16z^2 = 48$.

- We use Lagrange multipliers.
- The system is $yz = 2\lambda x$, $xz = 8\lambda y$, $xy = 32\lambda z$, $x^2 + 4y^2 + 16z^2 = 48$.
- If one variable is zero then so must be a second, so we get points $(x, y, z) = (\pm\sqrt{48}, 0, 0)$, $(0, \pm\sqrt{12}, 0)$, $(0, 0, \pm\sqrt{3})$.
- If no variables are zero, dividing the first two equations gives $y/x = x/(4y)$ so $x^2 = 4y^2$.
- Similarly, dividing the 1st and 3rd equations gives $x^2 = 16z^2$.
- So the last equation yields $3x^2 = 48$, so $(x, y, z) = (\pm 4, \pm 2, \pm 1)$ with all possible sign choices.
- The minimum is -8 (at 4 of these) and the maximum is 8 (at the other 4).

Review Problems, XIII

(#11f) Set up the integral of $\sqrt{x^2 + y^2 + z^2}$ on the region below $z = -3\sqrt{x^2 + y^2}$ and inside $x^2 + y^2 + z^2 = 4$.

Review Problems, XIII

(#11f) Set up the integral of $\sqrt{x^2+y^2+z^2}$ on the region below $z = -3\sqrt{x^2+y^2}$ and inside $x^2+y^2+z^2 = 4$.

- We can use any coordinate system, but here spherical will be the simplest.
- The surfaces are $\varphi = 5\pi/6$ and $\rho = 2$. There is no restriction on θ so it goes from 0 to 2π. We want the part below the cone, so φ goes from $5\pi/6$ to π, and ρ goes from 0 to 2.
- The function is $\sqrt{x^2+y^2+z^2} = \rho$ and the differential is $dV = \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$.
- So the integral is $\int_0^{2\pi} \int_{5\pi/6}^{\pi} \int_0^2 \rho \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$.

Review Problems, XV

(#11e) Set up an integral for the volume of the region bounded by $z = 2x$, $z = 3x$, $y = 1$, $y = 2$, $x = y$, and $x = 2y$.

Review Problems, XV

(#11e) Set up an integral for the volume of the region bounded by $z = 2x$, $z = 3x$, $y = 1$, $y = 2$, $x = y$, and $x = 2y$.

- We can use any coordinate system, but here rectangular will be easiest. To compute volume, we integrate the function $f(x, y, z) = 1$ on the region.
- If we reorder the equations as $y = 1$, $y = 2$, $x = y$, $x = 2y$, $z = 2x$, $z = 3x$ we can see that they are giving us the limits of integration for the integration order with y on the outside, x in the middle, and z on the inside, which is the order $dz\,dx\,dy$.
- Thus, the integral is $\int_1^2 \int_y^{2y} \int_{2x}^{3x} 1\,dz\,dx\,dy$.

Review Problems, XVI

(#4b) Set up (but do not evaluate) $\iint_R (x+y)\,dA$ on the region R between the curves $y = 8\sqrt{x}$ and $y = x^2$ using both integration orders $dy\,dx$ and $dx\,dy$.

Review Problems, XVI

(#4b) Set up (but do not evaluate) $\iint_R (x+y)\,dA$ on the region R between the curves $y = 8\sqrt{x}$ and $y = x^2$ using both integration orders $dy\,dx$ and $dx\,dy$.

- The curves intersect at $(0,0)$ and $(4,32)$.
- For $dy\,dx$ the ranges are $0 \le x \le 4$ and $x^2 \le y \le 8\sqrt{2}$, so the integral is $\displaystyle\int_0^4 \int_{x^2}^{8\sqrt{x}} (x+y)\,dy\,dx$.
- For $dx\,dy$ the ranges are $0 \le y \le 32$ and $y^2/64 \le x \le \sqrt{y}$ so the integral is $\displaystyle\int_0^{16} \int_{y^2/64}^{\sqrt{y}} (x+y)\,dy\,dx$.

Review Problems, XVII

(#5b) Reverse the order of integration for $\int_1^2 \int_y^{y^2} y^4 \, dx \, dy$.

Review Problems, XVII

(#5b) Reverse the order of integration for $\int_1^2 \int_y^{y^2} y^4 \, dx \, dy$.

- The region is defined by the inequalities $1 \leq x \leq 2$, $y \leq x \leq y^2$.
- From a quick sketch, we can see that when we change the order, we need to split into two pieces because the right curve changes from $y = x$ to $y = 2$ at $x = 2$.
- Thus, the new integral is $\int_1^2 \int_{\sqrt{x}}^{x} y^4 \, dy \, dx + \int_2^4 \int_{\sqrt{x}}^{2} y^4 \, dy \, dx$.

Review Problems, XVIII

(#1c) Find the minimum and maximum values of $f(x, y) = xy$, and all points where they occur, subject to $3x + y = 60$.

Review Problems, XVIII

(#1c) Find the minimum and maximum values of $f(x, y) = xy$, and all points where they occur, subject to $3x + y = 60$.

- We use Lagrange multipliers with $f = xy$ and $g = 3x + y$.
- The system is $y = 3\lambda$, $x = \lambda$, $3x + y = 60$.
- Thus, $6\lambda = 60$ so $\lambda = 10$.
- This yields a single point $(x, y) = (10, 30)$.
- Here, because the constraint region is not bounded, we have to worry about what happens as (x, y) goes to ∞. In that case, f goes to $-\infty$ for large x or large y.
- Thus, the minimum does not exist, whereas the maximum is 300 at $(10, 30)$.

Review Problems, XIX

(#11g) Set up a triple integral for the volume of the solid below $z = 5 - x^2 - y^2$, above the xy-plane, and outside $x^2 + y^2 = 1$.

Review Problems, XIX

(#11g) Set up a triple integral for the volume of the solid below $z = 5 - x^2 - y^2$, above the xy-plane, and outside $x^2 + y^2 = 1$.

- Here, cylindrical is the best choice since the bounding surfaces are $z = 5 - r^2$, $z = 0$, and $r = 1$.
- Here we have no restrictions on θ, and $1 \le r \le \sqrt{5}$ based on the surface intersections.
- Then z ranges from 0 to $5 - r^2$.
- The function is 1 for finding a volume, and the differential is $dV = r\,dz\,dr\,d\theta$.
- So the integral is $\int_0^{2\pi} \int_1^{\sqrt{5}} \int_0^{5-r^2} 1 \cdot r\,dz\,dr\,d\theta$.

Review Problems, XX

(#11a) Set up the integral $\iiint_D (x^2 + y^2)\, dV$ on the region D above $z = x^2 + y^2$, below $z = 7$, for $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Review Problems, XX

(#11a) Set up the integral $\iiint_D (x^2 + y^2)\, dV$ on the region D above $z = x^2 + y^2$, below $z = 7$, for $0 \le x \le 1$ and $0 \le y \le 2$.

- We can use any coordinate system, but here rectangular will be easiest.
- With order $dz\, dy\, dx$ the limits are $0 \le x \le 1$, $0 \le y \le 2$, and $x^2 + y^2 \le z \le 7$.
- Thus, the integral is $\displaystyle\int_0^1 \int_0^2 \int_{x^2+y^2}^7 (x^2 + y^2)\, dz\, dy\, dx$.

Summary

We did some more review problems for midterm 2.

Next lecture: Surfaces and surface integrals.