Math 2321 (Multivariable Calculus)

Lecture #24 of 37 $\sim$ March 17, 2021

Midterm #2 Review #1

Midterm 2 Exam Topics

The topics for the exam are as follows:

- Lagrange multipliers
- Double integrals in rectangular coordinates
- Changing the order of integration
- Double integrals in polar coordinates
- Triple integrals in rectangular coordinates
- Triple integrals in cylindrical coordinates
- Triple integrals in spherical coordinates
- Areas, volumes, average value, mass, center of mass

This represents § 2.6 + § 3.1−3.2 + § 3.3.2−§3.4 from the notes and WeBWorKs 5-8. Note that general changes of coordinates (§3.3.1) are NOT included.

Exam Information

The exam format is essentially the same as the first midterm.

- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- There are approximately 6 pages of material, about 1/5 multiple choice and the rest free response.
- I have set up a Piazza poll for you to select your desired exam window. Please make your selection by **this evening**. I will post your selection in Canvas so you can confirm it tomorrow.
- The "official" exam time limit is $65+25 = 90$ minutes, plus 30 minutes of turnaround time (not to be used for working).

Collaboration of any kind is not allowed. You may not discuss anything about the exam with anyone other than me (the instructor) until 5pm Eastern on Tuesday, March 23rd. This includes Piazza posts.

Review Problems, I

(#1b) Find the minimum and maximum values of $f(x, y) = xy^2$ subject to $x^2 + y^2 = 12$.

Review Problems, I

(#1b) Find the minimum and maximum values of $f(x, y) = xy^2$ subject to $x^2 + y^2 = 12$.

- We use Lagrange multipliers: here $f = xy^2$ and $g = x^2 + y^2$.
- The system is $\nabla f = \lambda \nabla g$ and $g = c$.
- Note $\nabla f = \langle y^2, 2xy \rangle$ and $\nabla g = \langle 2x, 2y \rangle$.
- So we get $y^2 = 2\lambda x$, $2xy = 2\lambda y$, $x^2 + y^2 = 12$.
- The second equation gives $y = 0$ or $x = \lambda$.
- If $y = 0$ then we get points $(\pm\sqrt{12}, 0)$.
- If $x = \lambda$ then the first equation gives $y^2 = 2\lambda^2$.
- Plugging into the third equation then yields $3\lambda^2 = 12$ so $\lambda = \pm 2$ and thus $(x, y) = (\pm 2, \pm\sqrt{8})$.
- Minimum is -16 at $(-2, \pm\sqrt{8})$, maximum is 16 at $(2, \pm\sqrt{8})$.

Review Problems, II

(#8) Evaluate the double integral $\int_0^8 \int_{x/2}^4 \frac{e^y}{y}\, dy\, dx$ by reversing the order of integration.

Review Problems, II

(#8) Evaluate the double integral $\int_0^8 \int_{x/2}^4 \frac{e^y}{y} \, dy \, dx$ by reversing the order of integration.

- The region is interior of a right triangle bounded by $x = 0$, $x = 8$, $y = x/2$, and $y = 4$, so it has vertices $(0, 0)$, $(8, 4)$, and $(0, 4)$.
- Thus, with order $dx\, dy$, we see that y ranges from 0 to 4, and then x ranges from 0 to $2y$.
- The reversed integral is then $\int_0^4 \int_0^{2y} \frac{e^y}{y} \, dx \, dy$.
- Evaluating yields $\int_0^4 \int_0^{2y} \frac{e^y}{y} \, dx \, dy = \int_0^4 2e^y \, dy = 2(e^4 - 1)$.

Review Problems, III

(#11d) Set up the integral of $z\sqrt{x^2+y^2}$ on the region with $x \leq 0$, inside $x^2+y^2=4$, above $z=0$, below $y+z=4$.

Review Problems, III

(#11d) Set up the integral of $z\sqrt{x^2+y^2}$ on the region with $x \le 0$, inside $x^2+y^2=4$, above $z=0$, below $y+z=4$.

- We can use any coordinate system, but here cylindrical will be easiest.
- The surfaces are $r=2$, $z=0$, and $z=4-r\sin\theta$.
- The restriction $x \le 0$ gives $\pi/2 \le \theta \le 3\pi/2$, and then r ranges from 0 to 2 and z ranges from 0 to $4-r\sin\theta$.
- The function is $z\sqrt{x^2+y^2} = zr$ and the differential is $dV = r\,dz\,dr\,d\theta$.
- So the integral is $\displaystyle\int_{\pi/2}^{3\pi/2}\int_0^2\int_0^{4-r\sin\theta} zr \cdot r\,dz\,dr\,d\theta$.

Review Problems, IV

(#4c) Set up (but do not evaluate) double integrals for the volume under $z = x^3$ above the triangle in the xy-plane with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$, using both integration orders $dy\,dx$ and $dx\,dy$.

Review Problems, IV

(#4c) Set up (but do not evaluate) double integrals for the volume under $z = x^3$ above the triangle in the xy-plane with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$, using both integration orders $dy\,dx$ and $dx\,dy$.

- A quick sketch of the triangle shows that the bounding lines are $y = x$ and $y = 2 - x$.
- For $dy\,dx$ we must split into two ranges for x, since the upper curve changes from $y = x$ to $y = 2 - x$ at $x = 1$. The result is $\displaystyle\int_0^1 \int_0^x x^3\,dy\,dx + \int_1^2 \int_0^{2-x} x^3\,dy\,dx$.
- For $dx\,dy$ we do not need to split, since the left curve is always $x = y$ and the right curve is always $x = 2 - y$. So the integral is $\displaystyle\int_0^1 \int_y^{2-y} x^3\,dx\,dy$.

Review Problems, V

(#1a) Find the minimum and maximum values of $f(x, y) = x + 3y + 2$, and all points where they occur, subject to the constraint $x^2 + y^2 = 40$.

Review Problems, V

(#1a) Find the minimum and maximum values of $f(x, y) = x + 3y + 2$, and all points where they occur, subject to the constraint $x^2 + y^2 = 40$.

- We use Lagrange multipliers with $f = x + 3y + 2$ and $g = x^2 + y^2$.
- The system is $\nabla f = \lambda \nabla g$ and $g = c$.
- Explicitly, this yields $1 = 2\lambda x$, $3 = 2\lambda y$, $x^2 + y^2 = 40$.
- The first two equations give $x = 1/(2\lambda)$, $y = 3/(2\lambda)$.
- Plugging into the third equation yields $10/(4\lambda^2) = 40$ so $\lambda = \pm 1/4$, yielding $(x, y) = (2, 6), (-2, -6)$.
- The minimum is -18, at $(-2, -6)$, and the maximum is 22, at $(2, 6)$.

Review Problems, VI

(#11h) Set up an integral for the average value of $x^2 + y^2 + z^2$ on the portion of $x^2 + y^2 + z^2 \le 4$ inside the first octant (with $x, y, z \ge 0$), which has volume $4\pi/3$.

Review Problems, VI

(#11h) Set up an integral for the average value of $x^2 + y^2 + z^2$ on the portion of $x^2 + y^2 + z^2 \leq 4$ inside the first octant (with $x, y, z \geq 0$), which has volume $4\pi/3$.

- We can use any coordinate system, but here spherical will be the simplest.
- The integration bounds are θ from 0 to $\pi/2$ ($x, y \geq 0$), φ from 0 to $\pi/2$ ($z \geq 0$), and ρ from 0 to 2.
- The function is $x^2 + y^2 + z^2 = \rho^2$ and the differential is $dV = \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$.
- So the average value is
$$\frac{1}{4\pi/3} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \rho^2 \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Review Problems, VII

(#7) Evaluate the double integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$ by converting it to polar coordinates.

Review Problems, VII

(#7) Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$ by converting it to polar coordinates.

- The region is defined by the inequalities $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$.
- This represents a quarter-circle: specifically, it is the interior of $x^2 + y^2 = 1$ in the first quadrant.
- In polar, the limits are $\theta = 0$ to $\theta = \pi/2$ and $r = 0$ to $r = 1$.
- The function is $\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$, and the differential is $dA = r\, dr\, d\theta$.
- So the polar integral is $\int_0^{\pi/2} \int_0^1 \frac{1}{r} \cdot r\, dr\, d\theta$.
- Evaluating gives $\int_0^{\pi/2} \int_0^1 \frac{1}{r} \cdot r\, dr\, d\theta = \int_0^{\pi/2} \int_0^1 1\, dr\, d\theta = \int_0^{\pi/2} 1\, d\theta = \pi/2$.

Review Problems, VIII

(#4a) Set up (but do not evaluate) the integral of x^2y on the region $0 \le x \le 1$, $0 \le y \le 3$ using both integration orders $dy\,dx$ and $dx\,dy$.

Review Problems, VIII

(#4a) Set up (but do not evaluate) the integral of $x^2 y$ on the region $0 \le x \le 1$, $0 \le y \le 3$ using both integration orders $dy\,dx$ and $dx\,dy$.

- The region is a rectangle.
- So with order $dy\,dx$ the integral is $\int_0^1 \int_0^3 x^2 y\,dy\,dx$.
- With order $dx\,dy$ the integral is $\int_0^3 \int_0^1 x^2 y\,dx\,dy$.

Review Problems, IX

(#11c) Set up the triple integral $\iiint_D (x^2 + y^2 + z^2)\, dV$ on the region D above $z = 2\sqrt{x^2 + y^2}$ and below $z = 3$.

Review Problems, IX

(#11c) Set up the triple integral $\iiint_D (x^2 + y^2 + z^2)\, dV$ on the region D above $z = 2\sqrt{x^2 + y^2}$ and below $z = 3$.

- Here, cylindrical is the best choice, since the region and function are easy to describe in cylindrical.
- The bounding surfaces are $z = 2r$ and $z = 3$, so there are no restrictions on θ. We also need $2r \leq 3$, so r ranges from 0 to $3/2$, and then z ranges from $2r$ to 3.
- The function is $x^2 + y^2 + z^2 = r^2 + z^2$ with differential $dV = r\, dz\, dr\, d\theta$.
- So the integral is $\int_0^{2\pi} \int_0^{3/2} \int_{2r}^{3} (r^2 + z^2)\, r\, dz\, dr\, d\theta$.

Review Problems, X

(#12c) Find the total mass, and the center of mass, of the solid between $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 3$ with density $d(x, y, z) = 3\sqrt{x^2 + y^2 + z^2}^{3/2}$ kg/m^3.

Review Problems, X

(#12c) Find the total mass, and the center of mass, of the solid between $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 3$ with density $d(x, y, z) = 3\sqrt{x^2 + y^2 + z^2}^{3/2}$ kg/m^3.

- We use spherical coordinates. The two surfaces are $\rho = \sqrt{2}$ and $\rho = \sqrt{3}$ while the density is $3\rho^3$ kg/m^3.
- Therefore, the mass is $M = \iiint_D d(x, y, z)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_{\sqrt{2}}^{\sqrt{3}} 3\rho^3 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.
- Evaluating the integral yields
$$M = \int_0^{2\pi} \int_0^{\pi} \frac{1}{2}\rho^6 \sin\varphi \Big|_{\rho=\sqrt{2}}^{\rho=\sqrt{3}} d\varphi \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi} \frac{19}{2} \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} 19 \, d\theta = 38\pi \text{ kg}.$$
- Since the solid is spherically symmetric, the center of mass is at the origin.

Review Problems, XI

(#11b) Set up a triple integral of xyz on the region above $z = y^2$, below $z = 9$, between $x = 1$ and $x = 2$.

Review Problems, XI

(#11b) Set up a triple integral of xyz on the region above $z = y^2$, below $z = 9$, between $x = 1$ and $x = 2$.

- Here, rectangular is the best choice.
- With order $dz\,dy\,dx$, we are given that x ranges from 1 to 2.
- For the y-limits, note that $z = y^2$ intersects $z = 9$ when $y = \pm 3$, so the y range is -3 to 3.
- Then z ranges from y^2 to 9.
- The function is xyz and the differential is $dz\,dy\,dx$.
- So the integral is $\int_1^2 \int_{-3}^3 \int_{y^2}^9 xyz\,dz\,dy\,dx$.

Review Problems, XII

(#4d) Set up (but do not evaluate) double integrals for the area of the region between the curves $y = x^2 - 1$ and $y = 5$ using both integration orders $dy\,dx$ and $dx\,dy$.

Review Problems, XII

(#4d) Set up (but do not evaluate) double integrals for the area of the region between the curves $y = x^2 - 1$ and $y = 5$ using both integration orders $dy\,dx$ and $dx\,dy$.

- To compute area, we integrate $f(x, y) = 1$ on the region.
- The curves intersect when $5 = x^2 - 1$ so that $x = \pm\sqrt{6}$, yielding points $(-\sqrt{6}, 5)$ and $(\sqrt{6}, 5)$. The vertex of the parabola is also $(0, -1)$.
- For $dy\,dx$ we can see that x ranges from $-\sqrt{6}$ to $\sqrt{6}$, and on this range, the lower curve is $y = x^2 - 1$ and the upper curve is $y = 5$. Thus the integral is $\int_{-\sqrt{6}}^{\sqrt{6}} \int_{x^2-1}^{5} 1\,dy\,dx$.
- For $dx\,dy$ we can see that y ranges from -1 to 5, and on this range, the left curve is $x = -\sqrt{y+1}$ and the right curve is $x = \sqrt{y+1}$. Thus the integral is $\int_{-1}^{5} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} 1\,dx\,dy$.

Review Problems, Lucky XIII

(#10b) Find $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ by converting to cylindrical or spherical.

Review Problems, Lucky XIII

(#10b) Find $\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$ by converting to cylindrical or spherical.

- Here, we want to use spherical, since the function and z-limits both involve spheres and ρ.
- The limits of integration indicate that the region is inside the sphere $x^2+y^2+z^2=4$. Specifically, it is the upper half (from the z limits) and also has $y \geq 0$, so it is the quarter-sphere where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq \pi/2$, with $0 \leq \rho \leq 2$.
- The differential is $dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$.
- So the integral is $\int_0^{\pi} \int_0^{\pi/2} \int_0^{2} \rho \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta =$

$$\int_0^{\pi} \int_0^{\pi/2} 4 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{\pi/2} 4\, d\theta = 2\pi.$$

Summary

We did some review problems for midterm 2.

Next lecture: Review for Midterm 2 (part 2)