Math 2321 (Multivariable Calculus)

Lecture #23 of 37 $\sim$ March 15, 2021

Line Integrals

- Center of Mass Examples
- Line Integrals
- Applications of Line Integrals

This material represents §4.1 from the course notes.

Exam Logistics

Please select your desired midterm 2 testing window on Piazza.

- The format of the exam will be similar to midterm 1.
- Midterm 2 covers sections 2.6 and chapter 3 of the course notes. This represents Lectures 14-22 and WeBWorKs 5-8.
- In particular, the material from Chapter 4 in today's lecture, on line integrals, is not on midterm 2. (Though it is conceptually quite similar, as you will see.)
- The lectures on Wednesday and Thursday will be devoted to exam review. I will go over problems from the review sheets, like with the midterm 1 reviews.

One More Moment(s problem), I

Since we didn't do many examples of center of mass calculations last time, let's do a few more! (This also counts as exam review because these are from the review sheet.)

- If we have a 2-dimensional plate of variable density $\delta(x, y)$ and total mass M, then the center of mass $(\bar{x}, \bar{y})$ has $\bar{x} = \dfrac{1}{M} \iint_R x\delta(x, y)\, dA$ and $\bar{y} = \dfrac{1}{M} \iint_R y\delta(x, y)\, dA$.
- If we have a 3-dimensional plate of variable density $\delta(x, y, z)$ and total mass M, then the center of mass $(\bar{x}, \bar{y}, \bar{z})$ has $\bar{x} = \dfrac{1}{M} \iiint_D x\delta(x, y, z)\, dV$, $\bar{y} = \dfrac{1}{M} \iiint_D y\delta(x, y, z)\, dV$, and $\bar{z} = \dfrac{1}{M} \iiint_D z\delta(x, y, z)\, dV$.

One More Moment(s problem), II

(#12a) Find the total mass and the center of mass for the solid bounded by $0\,\text{cm} \le x \le 1\,\text{cm}$, $0\,\text{cm} \le y \le 2\,\text{cm}$, and $0\,\text{cm} \le z \le 3\,\text{cm}$ with density $d(x, y, z) = z\,\text{g}/\text{cm}^3$.

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- Mass is $M = \iiint_D d(x,y,z)\,dV = \int_0^1 \int_0^2 \int_0^3 z\,dz\,dy\,dx = 9\,\text{g}$.
- The moments for the center of mass are
  $M_x = \iiint_D x\,d(x,y,z)\,dV = \int_0^1 \int_0^2 \int_0^3 xz\,dz\,dy\,dx = 9/2$,
  $M_y = \iiint_D y\,d(x,y,z)\,dV = \int_0^1 \int_0^2 \int_0^3 yz\,dz\,dy\,dx = 9$,
  $M_z = \iiint_D z\,d(x,y,z)\,dV = \int_0^1 \int_0^2 \int_0^3 z^2\,dz\,dy\,dx = 18$.
- So the center of mass is $\frac{1}{M}(M_x, M_y, M_z) = (\frac{1}{2}\,\text{cm}, 1\,\text{cm}, 2\,\text{cm})$.

One More Moment(s problem), III

(#12d) Find the total mass of the solid between $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{3(x^2 + y^2)}$ inside $x^2 + y^2 + z^2 = 9$ with density $d(x, y, z) = 1\,\text{mg}/\text{mm}^3$.

One More Moment(s problem), III

(#12d) Find the total mass of the solid between $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{3(x^2 + y^2)}$ inside $x^2 + y^2 + z^2 = 9$ with density $d(x, y, z) = 1\,\text{mg}/\text{mm}^3$.

- We use spherical coordinates. The surfaces are $\varphi = \pi/4$, $\varphi = \pi/6$, and $\rho = 3$, while the density is $d = 1$.
- Therefore, the mass is $M = \iiint_D d(x, y, z)\, dV = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_0^3 1 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.
- Evaluating the integral yields $M =$

$$\int_0^{2\pi} \int_{\pi/6}^{\pi/4} \frac{1}{3}\rho^3 \sin\varphi \Big|_{\rho=0}^{\rho=3} d\varphi \, d\theta = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} 9 \sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} 9(\sqrt{3} - 1)/2 \, d\theta = 9(\sqrt{3} - 1)\pi \text{ mg}.$$

§4: Vector Calculus, Overview

We now start the last chapter of the course (§4: Vector Calculus), which will combine elements from all three of the previous chapters. [To emphasize: this material is not on midterm 2.]

- We begin with two additional generalizations of integration: integration along curves (today) and integration on surfaces (next week).
- Then we discuss vector fields (vector-valued functions of several variables) and work, circulation, flux, integrals.
- We then study a number of generalizations of the Fundamental Theorem of Calculus that relate all of these different kinds of integrals to one another.
- Finally, we will discuss in detail some applications of all of these results to engineering and the physical sciences: physics, engineering, computer science, applied math, chemistry, etc.

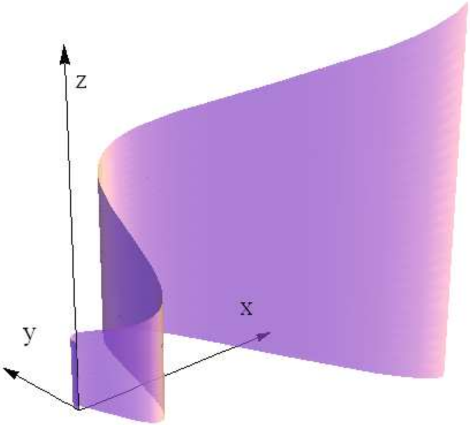
Line Integrals, I

The motivating problem for our development of double integrals was to find the volume under the graph of a surface.

- We will now develop yet another type of integral, called a line integral.
- The motivating problem is as follows: suppose we have a plane parametric curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and a function $f(x, y)$.
- If we "build a surface" along the curve with height given by the function $z = f(x, y)$, how can we calculate the area of this surface?
- This is a natural generalization of our typical single-variable integration problem, in which we build the "surface" inside a plane, thus making it the area under a curve.

Line Integrals, II

An illustration of the resulting "curtain surface", with $\mathbf{r}(t) = \langle t^2, \, t\cos(2\pi t)\rangle$, $f(x, y) = x + 1$, for $0 \le t \le 1.5$:



Line Integrals, III

Some other motivating questions that are analogues of the other applications of integration we have discussed:

- Given a parametric curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and a function $f(x, y, z)$, how can we calculate the average value of $f(x, y, z)$ on the curve?
- Given a thin wire shaped along a curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ with variable density $\delta(x, y)$, what is the wire's mass, and where is its center of mass?

As with the other integrals we have examined, we will use Riemann sums to formalize everything.

- The idea is to approximate the curve with straight line segments, sum (over all the segments) the function value times the length of the segment, and then take the limit as the segment lengths approach zero.

Line Integrals, IV

Definition

For a curve C, a <u>partition</u> of C into n pieces is a list of points (x_0, y_0), ... , (x_n, y_n) on C, with the nth segment having length $\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$.
The <u>norm</u> of the partition P is the largest number among all of the segment lengths in P.
If $f(x, y)$ is continuous, the <u>Riemann sum</u> of $f(x, y)$ on C corresponding to P is $RS_P(f) = \displaystyle\sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k$.

Now what we will do is take the limit of the Riemann sums as the size of the pieces goes to zero.

Line Integrals, V

Definition

For a function $f(x, y)$, we define
<u>the line integral of f on the curve C</u>, denoted $\int_C f(x, y)\, ds$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\mathrm{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.

- It can be proven (with significant effort) that, if $f(x, y)$ is continuous and the curve C is smooth, then a value of L satisfying the hypotheses actually does exist.
- The differential ds in the definition of the line integral is the "differential of arclength", which we discussed (way back!) in our study of vector-valued functions in Chapter 1.

Line Integrals, VI

In exactly the same way, we can use Riemann sums to give a formal definition of the line integral along a curve C in 3-space.

- Just go back to the previous two slides and put in the appropriate z terms everywhere!

Like with the other types of integrals, line integrals have a number of formal properties which can be deduced from the Riemann sum definition.

Line Integrals, VII

For an arbitrary constant D and continuous functions f and g, the following hold:

1. Integral of constant: $\int_C D\, ds = D \cdot \text{Arclength}(C)$.
2. Constant multiple of a function: $\int_C D\, f\, ds = D \cdot \int_C f\, ds$.
3. Addition of functions: $\int_C f\, ds + \int_C g\, ds = \int_C [f + g]\, ds$.
4. Subtraction of functions: $\int_C f\, ds - \int_C g\, ds = \int_C [f - g]\, ds$.
5. Nonnegativity: if $f \geq 0$, then $\int_C f\, ds \geq 0$.
6. Union: If C_1 and C_2 are curves such that C_2 starts where C_1 ends, and C is the curve obtained by gluing the curves end-to-end, then $\int_{C_1} f\, ds + \int_{C_2} f\, ds = \int_C f\, ds$.

Line Integrals, VIII

As usual, we will not actually use Riemann sums to compute line integrals. Instead, we will reduce them to "regular" single integrals.

Proposition (Line Integrals in the Plane)

If the curve C can be parametrized as $x = x(t)$, $y = y(t)$ for $a \leq t \leq b$, then $\displaystyle\int_C f(x,y)\,ds = \int_a^b f(x(t),\, y(t))\,\frac{ds}{dt}\,dt$, where $\dfrac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$ is the derivative of arclength.

The proof is to observe that the Riemann sum $\sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k$ for the line integral $\int_C f(x,y)\,ds$ is also a Riemann sum $\sum_{k=1}^{n} f(x_k, y_k)\,\frac{\Delta s_k}{\Delta t_k}\Delta t_k$ for the integral $\int_a^b f(x(t),\, y(t))\,\dfrac{ds}{dt}\,dt$.

Line Integrals, IX

We also have the 3-dimensional version, which is the same except with z terms:

Proposition (Line Integrals in 3-Space)

If the curve C can be parametrized as $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, then
$\displaystyle\int_C f(x, y, z)\, ds = \int_a^b f(x(t),\, y(t),\, z(t))\, \frac{ds}{dt}\, dt$, *where*
$\displaystyle\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ *is the derivative of arclength.*

Equivalently, this is the result of making a substitution in the integral by changing from s-coordinates to t-coordinates, where the differential changes using the rule $ds = \dfrac{ds}{dt} dt$.

Line Integrals, X

Thus, to evaluate the line integral of f on the curve C (i.e., the line integral $\int_C f(x, y, z)\, ds$), follow these steps:

1. Parametrize the curve C as a function of t, as $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$ for $a \le t \le b$.
2. Write the function f in terms of t: $f(x, y, z) = f(x(t), y(t), z(t))$.
3. Write the differential $ds = \dfrac{ds}{dt}\, dt = ||\mathbf{v}(t)||\ dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$ in terms of t.
4. Evaluate the resulting single-variable integral $\int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$.

Line Integrals, XI

Example: Integrate the function $f(x, y, z) = yz - 6x$ along the curve $\mathbf{r}(t) = \langle t^3, 6t, 3t^2 \rangle$ from $t = 0$ to $t = 1$.

Line Integrals, XI

<u>Example</u>: Integrate the function $f(x, y, z) = yz - 6x$ along the curve $\mathbf{r}(t) = \langle t^3, 6t, 3t^2 \rangle$ from $t = 0$ to $t = 1$.

- We have $f(x, y, z) = yz - 6x = (6t)(3t^2) - 6t^3 = 12t^3$.
- We also have
  $ds = \sqrt{(3t^2)^2 + (6)^2 + (6t)^2} = \sqrt{9t^4 + 36t^2 + 36} = 3t^2 + 6$.
- The integral is therefore
  $$\int_0^1 (12t^3)(3t^2 + 6)dt = \int_0^1 (36t^5 + 72t^3)\, dt = 24.$$

Line Integrals, XII

Example: Integrate the function $f(x, y, z) = z\sqrt{x^2 + y^2}$ along the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ from the point $(1, 0, 0)$ to the point $(1, 0, 4\pi)$.

Line Integrals, XII

Example: Integrate the function $f(x, y, z) = z\sqrt{x^2 + y^2}$ along the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ from the point $(1, 0, 0)$ to the point $(1, 0, 4\pi)$.

- We are given the parametrization, and we want the range $0 \leq t \leq 4\pi$.
- We have $f(x, y, z) = t\sqrt{\cos^2 t + \sin^2 t} = t$, and we also have $ds = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$.
- The integral is therefore $\displaystyle\int_0^{4\pi} t\sqrt{2}dt = 8\pi^2\sqrt{2}$.

Line Integrals, XIII

<u>Example</u>: Integrate the function $f(x, y) = x^2 + y$ along the top half of the unit circle $x^2 + y^2 = 1$, starting at $(1, 0)$ and ending at $(-1, 0)$.

Line Integrals, XIII

<u>Example</u>: Integrate the function $f(x, y) = x^2 + y$ along the top half of the unit circle $x^2 + y^2 = 1$, starting at $(1, 0)$ and ending at $(-1, 0)$.

- First, we need to parametrize the curve.

Line Integrals, XIII

Example: Integrate the function $f(x, y) = x^2 + y$ along the top half of the unit circle $x^2 + y^2 = 1$, starting at $(1, 0)$ and ending at $(-1, 0)$.

- First, we need to parametrize the curve.
- The unit circle is parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, and the range we want is $0 \leq t \leq \pi$.
- We have $f(x, y) = x^2 + y = \cos^2 t + \sin t$, and we also have $ds = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.
- The integral is therefore $\displaystyle\int_0^\pi \left[\cos^2 t + \sin t\right] dt = \int_0^\pi \left[\frac{1 + \cos 2t}{2} + \sin t\right] dt = \frac{\pi}{2} + 2.$

Line Integrals, XIV

To find the average value of a function on a curve, we simply integrate the function over the curve, and then divide by the curve's arclength.

- This is the same procedure we used for finding the average value of a function on a region: integrate the function, and then divide by the size of the region.
- To find the arclength of a curve C, we can integrate the function 1 along the curve.

Line Integrals, XV

Example: Let C be the line segment from $(1, -1, 0)$ to $(2, 2, 1)$.

1. Set up a parametrization of C and use it to find the arclength of C.

Line Integrals, XV

Example: Let C be the line segment from $(1, -1, 0)$ to $(2, 2, 1)$.

1. Set up a parametrization of C and use it to find the arclength of C.

- The direction vector for the line is $\mathbf{v} = \langle 2, 2, 1\rangle - \langle 1, -1, 0\rangle = \langle 1, 3, 1\rangle$. Thus, we can parametrize the line segment as $\langle x, y, z\rangle = \langle 1, -1, 0\rangle + t\,\langle 1, 3, 1\rangle$ for $0 \le t \le 1$.
- Explicitly, $x = 1 + t$, $y = -1 + 3t$, $z = t$ for $0 \le t \le 1$.
- Then $\dfrac{ds}{dt} = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$, so the arclength is $\int_0^1 1\,ds = \int_0^1 \sqrt{11}dt = \sqrt{11}$ (which we could also have found using the distance formula).

Line Integrals, XVI

Example: Let C be the line segment from $(1, -1, 0)$ to $(2, 2, 1)$.

2. Find the average value of $f(x, y, z) = x^2 + y^2 + z^2$ on C.

Line Integrals, XVI

Example: Let C be the line segment from $(1, -1, 0)$ to $(2, 2, 1)$.

2. Find the average value of $f(x, y, z) = x^2 + y^2 + z^2$ on C.

- We found the arclength was $\sqrt{11}$. Now we set up the integral of the function.
- The function is $f(x, y, z) = x^2 + y^2 + z^2 = (1 + t)^2 + (-1 + 3t)^2 + (t)^2 = 11t^2 - 4t + 2$.
- Since $x'(t) = 1$, $y'(t) = 3$, and $z'(t) = 1$, we also have
- The integral of f is therefore $\int_0^1 \left[11t^2 - 4t + 2\right] \sqrt{11}dt = \sqrt{11}\left[\frac{11}{3}t^3 - 2t^2 + 2t\right]\Big|_{t=0}^{1} = \frac{11\sqrt{11}}{3}$.
- To compute the average value, we divide by the arclength, giving an average of $11/3$.

Line Integrals, XVII

We also have formulas for the mass and moments of a wire of variable density:

Center of Mass and Moment Formulas (Thin Wire): Given a 1-dimensional wire of variable density $\delta(x, y, z)$ along a parametric curve C in 3-space:

- The total mass M is given by $M = \int_C \delta(x, y, z)\, ds$.
- The x-moment M_{yz} is given by $M_{yz} = \int_C x\, \delta(x, y, z)\, ds$.
- The y-moment M_{xz} is given by $M_{xz} = \int_C y\, \delta(x, y, z)\, ds$.
- The z-moment M_{xy} is given by $M_{xy} = \int_C z\, \delta(x, y, z)\, ds$.
- The center of mass $(\bar{x}, \bar{y}, \bar{z})$ is $\left(\dfrac{M_{yz}}{M}, \dfrac{M_{xz}}{M}, \dfrac{M_{xy}}{M}\right)$.
- Note: For a wire in 2-space, the formulas are essentially the same (except without the z-coordinate), though the x-moment is denoted M_y and the y-moment is denoted M_x.

Line Integrals, XVIII

Example: Find the total mass, and the center of mass, of a thin wire in the xy-plane having the shape of the unit circle with variable density $\delta(x, y) = 2 + x$.

Line Integrals, XVIII

<u>Example</u>: Find the total mass, and the center of mass, of a thin wire in the xy-plane having the shape of the unit circle with variable density $\delta(x, y) = 2 + x$.

- We can parametrize the unit circle with $x = \cos t$, $y = \sin t$, so $\dfrac{ds}{dt} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$.
- The total mass is $M = \int_C \delta(x, y)\, ds = \int_0^{2\pi} (2 + \cos t)\, dt = 2\pi$.
- The x-moment M_y is $M_y = \int_C x\, \delta(x, y)\, ds =$ $\int_0^{2\pi} \cos t(2 + \cos t)\, dt = \left[2 \sin t + \dfrac{1}{2} t + \dfrac{1}{4} \sin(2t)\right]\Big|_{t=0}^{2\pi} = \pi$.
- The y-moment M_x is $M_x = \int_C y\, \delta(x, y)\, ds =$ $\int_0^{2\pi} \sin t(2 + \cos t)\, dt = \left[-2 \cos t - \dfrac{1}{4} \cos(2t)\right]\Big|_{t=0}^{2\pi} = 0$.
- So the center of mass is $\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{1}{2}, 0\right)$.

Line Integrals, XIX

We will also be interested in computing line integrals involving the differentials dx, dy, and dz rather than ds: namely, expressions of the form $\displaystyle\int_C f\,dx + g\,dy + h\,dz$.

- We evaluate such line integrals by making the appropriate substitutions.
- Specifically, if C is parametrized by $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, then the line integral $\displaystyle\int_C f\,dx + g\,dy + h\,dz$ is given by the single-variable integral $\displaystyle\int_a^b \left[f\,\frac{dx}{dt} + g\,\frac{dy}{dt} + h\,\frac{dz}{dt} \right] dt$.

Line Integrals, XX

Example: Find $\int_C y\,dx + z\,dy + x^2\,dz$, where C is the curve $(x, y, z) = (t, t^2, t^3)$ ranging from $t = 0$ to $t = 1$.

Line Integrals, XX

Example: Find $\int_C y\,dx + z\,dy + x^2\,dz$, where C is the curve $(x, y, z) = (t, t^2, t^3)$ ranging from $t = 0$ to $t = 1$.

- We have $x = t$, $y = t^2$, and $z = t^3$.
- Thus, $dx = dt$, $dy = 2t\,dt$, and $dz = 3t^2\,dt$.
- The integral is $\int_0^1 \left[t^2 \cdot dt + 3t^2 \cdot 2t\,dt + t^2 \cdot 3t^2\,dt\right]$ $= \int_0^1 \left[t^2 + 6t^3 + 3t^4\right] dt = 73/30.$

Line Integrals, XXI

<u>Example</u>: Find $\int_C x\,dy - y\,dx$, where C is the upper half of the ellipse $x^2/9 + y^2/16 = 1$, starting at $(3, 0)$ and ending at $(-3, 0)$.

Line Integrals, XXI

Example: Find $\int_C x\,dy - y\,dx$, where C is the upper half of the ellipse $x^2/9 + y^2/16 = 1$, starting at $(3,0)$ and ending at $(-3,0)$.

- This ellipse is parametrized by $\mathbf{r}(t) = \langle 3\cos t,\ 4\sin t\rangle$, and the range we want is $0 \le t \le \pi$.
- We have $x = 3\cos t$ and $y = 4\sin t$, so that $dx = -3\sin t\,dt$ and $dy = 4\cos t\,dt$.
- The desired integral is
$$\int_0^{\pi} [3\cos t \cdot (4\cos t\,dt) - 4\sin t \cdot (-3\sin t\,dt)]$$
$$= \int_0^{\pi} \left[12\cos^2 t + 12\sin^2 t\right] dt = 12\pi.$$

Summary

We finished some examples about center of mass.

We developed line integrals and discussed how to set them up as one-variable integrals.

We discussed how to compute average values and masses on curves using line integrals.

Next lecture: Review for Midterm 2 (part 1)