Math 2321 (Multivariable Calculus)

Lecture #22 of 38 $\sim$ March 11, 2021

Applications of Double and Triple Integrals

- Areas, Volumes, and Average Values
- Masses, Moments, and Center of Mass

This material represents §3.4.1-3.4.2 from the course notes.

This lecture is the end of the material covered on Midterm 2.

Reminders, I

Recall cylindrical coordinates:

Definition

The cylindrical coordinates (r, θ, z) *of a point whose rectangular coordinates are* (x, y, z) *satisfy* $x = r\cos(\theta)$, $y = r\sin(\theta)$, *and* $z = z$ *for* $r \geq 0$ *and* $0 \leq \theta \leq 2\pi$.

- The main feature to remember is that $r = \sqrt{x^2 + y^2}$.
- The volume differential in cylindrical coordinates is
$$\boxed{dV = r\,dz\,dr\,d\theta}.$$

Reminders, II

Also recall spherical coordinates:

Definition

The *<u>spherical coordinates</u>* (ρ, θ, φ) *of a point* (x, y, z) *satisfy* $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$ *for* $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, *and* $0 \leq \varphi \leq \pi$.

- The angle θ measures longitude and is the same as in cylindrical, while the angle φ measures latitude, and ρ measures the distance to the origin.
- We also have $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$ and $\varphi = \tan^{-1}(r/z) = \tan^{-1}(\sqrt{x^2 + y^2} z)$.
- The differential in spherical coordinates is

$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}.$$

Areas, Volumes, and Averages, I

We can use multiple integrals to compute areas and volumes:

$$\begin{aligned} \text{Area}(R) &= \iint_R 1\, dA \\ \text{Volume}(D) &= \iiint_D 1\, dV. \end{aligned}$$

Closely related is the notion of the average value of a function:

Definition

The average value of f on a plane region R, and solid region D, are $\dfrac{1}{Area(R)} \iint_R f\, dA$ *and* $\dfrac{1}{Volume(D)} \iiint_D f\, dV$ *respectively.*

Areas, Volumes, and Averages, II

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

1. Set up a double integral for the volume of D in rectangular.
2. Set up a double integral for the volume of D in polar.
3. Set up a triple integral for the volume of D in rectangular.
4. Set up a triple integral for the volume of D in cylindrical.
5. Find the average value of z on D.

Areas, Volumes, and Averages, II

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3. Set up a triple integral for the volume of D in rectangular.
4. Set up a triple integral for the volume of D in cylindrical.
5. Find the average value of z on D.

- Although we are first asked for double integrals, it will be easier to work things out if we think in cylindrical coordinates, since both surfaces have nice descriptions in cylindrical.

Areas, Volumes, and Averages, III

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Areas, Volumes, and Averages, III

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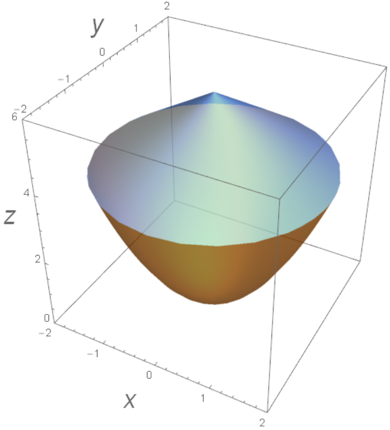

Areas, Volumes, and Averages, III

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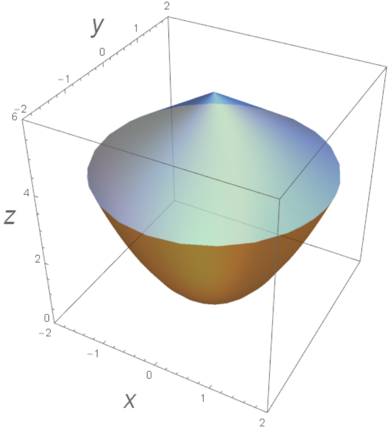

- D is below the graph of $z = 6 - \sqrt{x^2 + y^2}$ and above the graph of $z = x^2 + y^2$.
- In cylindrical, these are $z = 6 - r$ and $z = r^2$, so they intersect when $6 - r = r^2$, which is to say, when $r = 2$.
- So, the projection of D into the xy-plane is the interior of the circle $r = 2$.

Areas, Volumes, and Averages, IV

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

1. Set up a double integral for the volume of D in rectangular.

Areas, Volumes, and Averages, IV

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

1. Set up a double integral for the volume of D in rectangular.

- The volume is equal to the integral of the difference in heights between the bottom surface and the top surface.
- Since the projection of D into the xy-plane is the interior of the circle $r = 2$, which is $x^2 + y^2 = 4$ in rectangular, the volume of D is given by $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [6 - \sqrt{x^2 + y^2} - (x^2 + y^2)]\, dy\, dx.$$
- This one is not so nice to evaluate, since we really want to be using polar.

Areas, Volumes, and Averages, V

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

2. Set up a double integral for the volume of D in polar.

Areas, Volumes, and Averages, V

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

2. Set up a double integral for the volume of D in polar.

- The volume is equal to the integral of the difference in heights between the bottom surface and the top surface.
- Since the projection of D into the xy-plane is the interior of the circle $r = 2$, the volume of D in polar is given by $\int_0^{2\pi} \int_0^2 [6 - r - r^2] \cdot r \, dr \, d\theta$

Areas, Volumes, and Averages, V

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

2. Set up a double integral for the volume of D in polar.

- The volume is equal to the integral of the difference in heights between the bottom surface and the top surface.
- Since the projection of D into the xy-plane is the interior of the circle $r = 2$, the volume of D in polar is given by
$$\int_0^{2\pi} \int_0^2 [6 - r - r^2] \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [6r - r^2 - r^3] \, dr \, d\theta$$
$$= \int_0^{2\pi} [3r^2 - \frac{1}{3}r^3 - \frac{1}{4}r^4]\Big|_{r=0}^{2} \, d\theta = \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}.$$

Areas, Volumes, and Averages, VI

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

3. Set up a triple integral for the volume of D in rectangular.

Areas, Volumes, and Averages, VI

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

3. Set up a triple integral for the volume of D in rectangular.

- We use the rectangular region from the double integral (the interior of $x^2 + y^2 = 4$) for the x and y limits, while the z limits are $x^2 + y^2$ and $6 - \sqrt{x^2 + y^2}$.
- Thus, the volume integral is $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{6-\sqrt{x^2+y^2}} 1 \, dz \, dy \, dx$.
- Notice that if we evaluate the inner integral, we obtain the double integral $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [6 - \sqrt{x^2 + y^2} - (x^2 + y^2)] \, dy \, dx$ that we had earlier.

Areas, Volumes, and Averages, VII

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

4. Set up a triple integral for the volume of D in cylindrical.

Areas, Volumes, and Averages, VII

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

4. Set up a triple integral for the volume of D in cylindrical.

- We use the polar region from the double integral (the interior of $r = 2$) for the r and θ limits, while the z limits are r^2 and $6 - r$.
- Thus, the volume integral is $\int_0^{2\pi} \int_0^2 \int_{r^2}^{6-r} 1 \cdot r \, dz \, dr \, d\theta$.
- Like with the rectangular integral, if we evaluate the inner integral, we obtain the polar double integral $\int_0^{2\pi} \int_0^2 [6 - r - r^2] \cdot r \, dr \, d\theta$ that we had earlier.

Areas, Volumes, and Averages, VIII

<u>Example</u>: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

5. Find the average value of z on D.

Areas, Volumes, and Averages, VIII

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

5. Find the average value of z on D.

- We computed the volume of D as $\dfrac{32\pi}{3}$ earlier.
- To find the average value of z, we need to set up $\iiint_D z\, dV$.
- We use cylindrical, since it is clearly the most convenient.
- The integral is $\displaystyle\int_0^{2\pi} \int_0^{2} \int_{r^2}^{6-r} z \cdot r\, dz\, dr\, d\theta$

Areas, Volumes, and Averages, VIII

Example: Let D be the region between the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

5. Find the average value of z on D.

- We computed the volume of D as $\dfrac{32\pi}{3}$ earlier.
- To find the average value of z, we need to set up $\iiint_D z\, dV$.
- We use cylindrical, since it is clearly the most convenient.
- The integral is $\displaystyle\int_0^{2\pi}\int_0^2\int_{r^2}^{6-r} z \cdot r\, dz\, dr\, d\theta$

  $$= \int_0^{2\pi}\int_0^2 \tfrac{1}{2}[r(6-r)^2 - r^5]\, dr\, d\theta = \int_0^{2\pi} \frac{50}{3}\, d\theta = \frac{100\pi}{3}.$$
- Thus, the average value of z on D is $\dfrac{100\pi/3}{32\pi/3} = \dfrac{100}{32} = 3.125$.

Areas, Volumes, and Averages, VIII

Example: Find the average value of $(x^2 + y^2 + z^2)^{3/2}$ on the bottom half of the sphere of radius 3 centered at $(0, 0, 0)$.

Areas, Volumes, and Averages, VIII

Example: Find the average value of $(x^2+y^2+z^2)^{3/2}$ on the bottom half of the sphere of radius 3 centered at $(0,0,0)$.

- We want to set this up in spherical coordinates.
- In spherical, the region D is bounded by $0 \le \theta \le 2\pi$, $\pi/2 \le \varphi \le \pi$, and $0 \le \rho \le 3$.
- First, we need to compute the volume, which is
$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^3 1 \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$
$$= \int_0^{2\pi} \int_{\pi/2}^{\pi} 9 \sin(\varphi)\, d\varphi\, d\theta = \int_0^{2\pi} 9\, d\theta = 18\pi.$$
- Alternatively, we could just have used the formula for the volume of a sphere! (But that requires you to *know* it....)

Areas, Volumes, and Averages, IX

Example: Find the average value of $(x^2+y^2+z^2)^{3/2}$ on the bottom half of the sphere of radius 3 centered at $(0,0,0)$.

- The volume of the region is 18π.
- Then, since the function is $(x^2+y^2+z^2)^{3/2} = \rho^3$, the average value is equal to
  $$\frac{1}{18\pi}\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^3 \rho^3\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$
  $$=\frac{1}{18\pi}\int_0^{2\pi}\int_{\pi/2}^{\pi}\frac{243}{2}\sin(\varphi)\,d\varphi\,d\theta$$
  $$=\frac{1}{18\pi}\int_0^{2\pi}\frac{243}{2}\,d\theta=\frac{1}{18\pi}\cdot 243\pi=\frac{27}{2}=13.5.$$

Masses, Moments, and Center of Mass, I

Another application of double and triple integrals is to compute the mass of an object with a variable density.

- If we have a 2-dimensional plate of variable density $\delta(x, y)$ on a region R, then the total mass of the plate is $M = \iint_R \delta(x, y)\, dA$.
- Similarly, if we have a 3-dimensional solid of variable density $\delta(x, y, z)$ on a region D, then the total mass of the solid is $M = \iiint_D \delta(x, y, z)\, dV$.

Masses, Moments, and Center of Mass, II

Example: A triangular plate with vertices at $(0, 0)$, $(2, 0)$, and $(0, 1)$ has a variable density $\delta(x, y) = 1 + x$. Find the plate's mass.

Masses, Moments, and Center of Mass, II

Example: A triangular plate with vertices at $(0,0)$, $(2,0)$, and $(0,1)$ has a variable density $\delta(x,y) = 1+x$. Find the plate's mass.

- It is easiest here to use rectangular coordinates. Note that the line joining $(2,0)$ to $(0,1)$ has equation $x+2y=2$.
- With integration order $dy\,dx$, the desired mass integral is
$$\int_0^2 \int_0^{1-x/2} (1+x)\,dy\,dx = \int_0^2 y(1+x)\Big|_{y=0}^{1-x/2}\,dx$$
$$= \int_0^2 (1+x/2-x^2/2)\,dx = \frac{5}{3}.$$
- With integration order $dx\,dy$, the desired mass integral is
$$\int_0^1 \int_0^{2-2y} (1+x)\,dx\,dy = \int_0^1 (x+x^2/2)\Big|_{x=0}^{2-2y}\,dy$$
$$= \int_0^1 (4-6y+2y^2)\,dy = \frac{5}{3}.$$

Masses, Moments, and Center of Mass, III

<u>Example</u>: A solid sphere of radius 2 cm centered at the origin has a variable density: at a distance d cm away from the origin, its density is $(3 - d)\ \mathrm{g/cm^3}$. Find the mass of the sphere.

Masses, Moments, and Center of Mass, III

<u>Example</u>: A solid sphere of radius 2 cm centered at the origin has a variable density: at a distance d cm away from the origin, its density is $(3-d)\ \mathrm{g}/\mathrm{cm}^3$. Find the mass of the sphere.

- Because of the spherical symmetry, it will be most convenient to set up the mass integral in spherical coordinates.
- The density is $d = 3 - \rho$ in spherical coordinates, and the solid D has integration bounds $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, $0 \le \rho \le 2$.
- Thus, the mass is given by
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2}(3-\rho)\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi}4\sin(\varphi)\,d\varphi\,d\theta=\int_0^{2\pi}8\,d\theta=(16\pi)\,\mathrm{g}.$$

Masses, Moments, and Center of Mass, IV

Another important application of integration is to find the center of mass of an object.

- The center of mass of a physical object is its "balancing point", where, if the object is supported only at that point, gravity will not cause it to tip over.
- The center of mass is also called the centroid of an object.

Masses, Moments, and Center of Mass, V

To find the center of mass, recall the lever principle of Archimedes: if we place masses m_1 and m_2 at distances d_1 and d_2 from a pivot point, then the masses will balance when $m_1 d_1 = m_2 d_2$.

- More generally, if we have a mass m_i at x-coordinate x_i, where the pivot is at $x = 0$, then the masses will balance precisely when $\sum m_i x_i = 0$.
- If the balance point is at $x = \bar{x}$ instead, then the balance condition becomes $\sum m_i(x_i - \bar{x}) = 0$, so $\sum m_i x_i = \bar{x} \sum m_i$.
- If we instead have a continuous density function $\delta(x)$, then (by recognizing the sum above as a Riemann sum), then the corresponding balance condition becomes $\int_{-\infty}^{\infty} x\delta(x)\,dx = \bar{x} \int_{-\infty}^{\infty} \delta(x)\,dx$, so $\bar{x} = \dfrac{\int_{-\infty}^{\infty} x\delta(x)\,dx}{\int_{-\infty}^{\infty} \delta(x)\,dx}$.

Masses, Moments, and Center of Mass, VI

To summarize the previous slide, if we have a continuous density function $\delta(x)$ for masses on a line, then the center of mass $\bar{x}$ is given by $\bar{x} = \dfrac{\int_{-\infty}^{\infty} x\delta(x)\,dx}{\int_{-\infty}^{\infty} \delta(x)\,dx}$.

- Notice that the numerator is the integral of x times the density function, while the denominator is simply the total mass M.
- We can think of this formula as computing the average value of x on the line, weighted by the density function $\delta(x)$.

Masses, Moments, and Center of Mass, VII

We can now easily generalize these ideas into higher dimensions.

- If we have a 2-dimensional plate of variable density $\delta(x, y)$ and total mass M, then the center of mass $(\bar{x}, \bar{y})$ has $\bar{x} = \dfrac{1}{M} \iint_R x\delta(x, y)\, dA$ and $\bar{y} = \dfrac{1}{M} \iint_R y\delta(x, y)\, dA$.
- If we have a 3-dimensional plate of variable density $\delta(x, y, z)$ and total mass M, then the center of mass $(\bar{x}, \bar{y}, \bar{z})$ has $\bar{x} = \dfrac{1}{M} \iiint_D x\delta(x, y, z)\, dV$, $\bar{y} = \dfrac{1}{M} \iiint_D y\delta(x, y, z)\, dV$, and $\bar{z} = \dfrac{1}{M} \iiint_D z\delta(x, y, z)\, dV$.

Masses, Moments, and Center of Mass, VIII

The integrals we compute for the center of mass are called first moments: the first x-moment is the integral of x times the density function, the first y-moment is the integral of y times the density function, and so forth.

- Various higher moments also exist (e.g., the second moment of x is the integral of x^2 times the density function, etc.).
- One moment arising often in physics is the moment of inertia of an object to an axis: this is the second moment of the distance to the axis.
- The moment of inertia of an object relative to an axis through its center of mass measures how much torque is required to impart angular acceleration around that axis.
- In particular, an object with a high moment of inertia will roll more slowly down an incline than an object with a smaller moment of inertia.

Masses, Moments, and Center of Mass, IX

Example: Find the center of mass of the region with density $\delta = 1$ inside $x^2 + y^2 = 1$ in the first quadrant.

Masses, Moments, and Center of Mass, IX

Example: Find the center of mass of the region with density $\delta = 1$ inside $x^2 + y^2 = 1$ in the first quadrant.

- We need to compute $\iint_R x\, dA$ and $\iint_R y\, dA$, since the mass here is simply the area of the quarter-circle, which is $\pi/4$.
- We set up the integrals in polar coordinates.
- The x-moment is
$$\int_0^{\pi/2} \int_0^1 r \cos\theta \cdot r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{3} \cos\theta = \frac{1}{3}.$$
- The y-moment is
$$\int_0^{\pi/2} \int_0^1 r \sin\theta \cdot r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{3} \sin\theta = \frac{1}{3}.$$
- Thus, the center of mass is $(\bar{x}, \bar{y}) = (\frac{4}{3\pi}, \frac{4}{3\pi})$.
- We could have used the symmetry of the quarter-circle to observe that the x and y coordinates of the center of mass would be equal.

Summary

We discussed how to use double and triple integrals to compute areas, volumes, and average values.

We discussed how to use double and triple integrals to compute masses, moments, and the center of mass of an object.

Next lecture: Line integrals.