Math 2321 (Multivariable Calculus)

Lecture #21 of 38 $\sim$ March 10, 2020

Triple Integrals in Cylindrical and Spherical Coordinates

- Triple Integrals in Cylindrical and Spherical Coordinates
- Areas, Volumes, and Average Values

This material represents §3.3.4-3.4.1 from the course notes.

Reminders, I

Recall cylindrical coordinates:

Definition

The *cylindrical coordinates* (r, θ, z) *of a point whose rectangular coordinates are* (x, y, z) *satisfy* $x = r\cos(\theta)$, $y = r\sin(\theta)$, *and* $z = z$ *for* $r \geq 0$ *and* $0 \leq \theta \leq 2\pi$.

- The main feature to remember is that $r = \sqrt{x^2 + y^2}$.
- The volume differential in cylindrical coordinates is

$$\boxed{dV = r\,dz\,dr\,d\theta}.$$

Reminders, II

Also recall spherical coordinates:

Definition

The <u>*spherical coordinates*</u> (ρ, θ, φ) *of a point* (x, y, z) *satisfy* $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$ *for* $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, *and* $0 \leq \varphi \leq \pi$.

- The angle θ measures longitude and is the same as in cylindrical, while the angle φ measures latitude, and ρ measures the distance to the origin.
- We also have $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$ and $\varphi = \tan^{-1}(r/z) = \tan^{-1}(\sqrt{x^2 + y^2}/z)$.
- The differential in spherical coordinates is

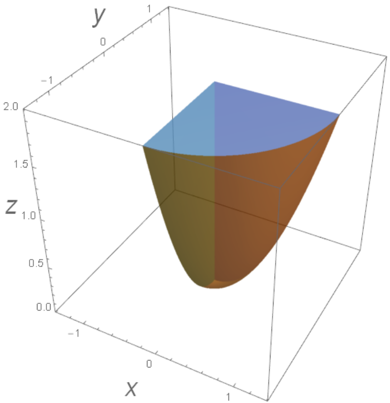
$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}.$$

Cylindrical and Spherical, I

Example: Evaluate $\iiint_D z^2 \, dV$ where D is the region between the surfaces $z = x^2 + y^2$ and $z = 2$ where $x \geq 0$ and $y \leq 0$.

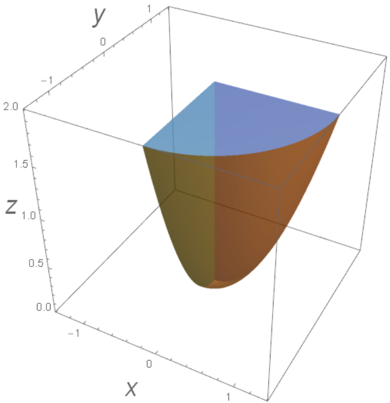
Cylindrical and Spherical, I

<u>Example</u>: Evaluate $\iiint_D z^2\,dV$ where D is the region between the surfaces $z = x^2 + y^2$ and $z = 2$ where $x \geq 0$ and $y \leq 0$.



Cylindrical and Spherical, I

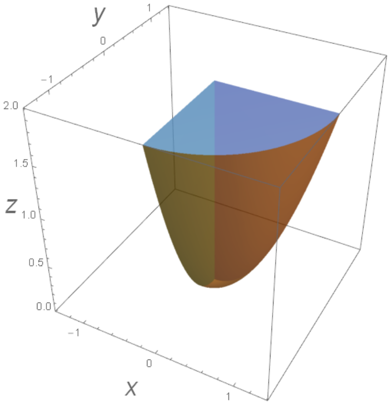
Example: Evaluate $\iiint_D z^2\,dV$ where D is the region between the surfaces $z = x^2 + y^2$ and $z = 2$ where $x \geq 0$ and $y \leq 0$.



- We set up in cylindrical.
- In cylindrical coordinates, the bounding surfaces are $z = r^2$ and $z = 2$, and the restrictions on x and y tells us that $3\pi/2 \leq \theta \leq 2\pi$.
- Note that $z = r^2$ intersects $z = 2$ when $r = \sqrt{2}$. So, our range for r is $0 \leq r \leq \sqrt{2}$, and then the bounds for z are $z = r^2$ and $z = 2$.

Cylindrical and Spherical, II

<u>Example</u>: Evaluate $\iiint_D z^2\,dV$ where D is the region between the surfaces $z = x^2 + y^2$ and $z = 2$ where $x \geq 0$ and $y \leq 0$.



- The function is $f(x,y,z) = z^2$, and the differential is $dV = r\,dz\,dr\,d\theta$.
- Thus, the integral is

$$\int_{3\pi/2}^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{2} z^2 r\,dz\,dr\,d\theta$$

$$= \int_{3\pi/2}^{2\pi}\int_0^{\sqrt{2}} \tfrac{1}{3}(8r - r^7)dr\,d\theta$$

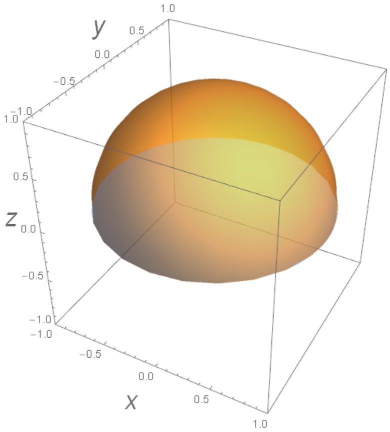
$$= \int_{3\pi/2}^{2\pi} 2\,d\theta = \pi.$$

Cylindrical and Spherical, III

Example: Integrate the function $f(x, y, z) = \frac{z}{\sqrt{x^2+y^2}}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

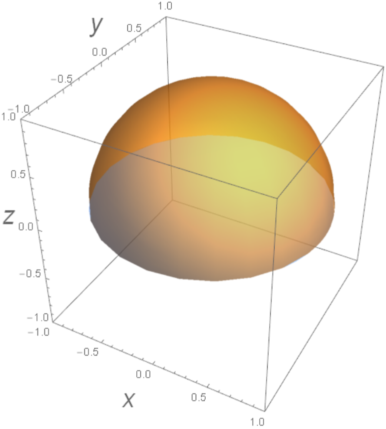
Cylindrical and Spherical, III

Example: Integrate the function $f(x, y, z) = \frac{z}{\sqrt{x^2+y^2}}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$.



Cylindrical and Spherical, III

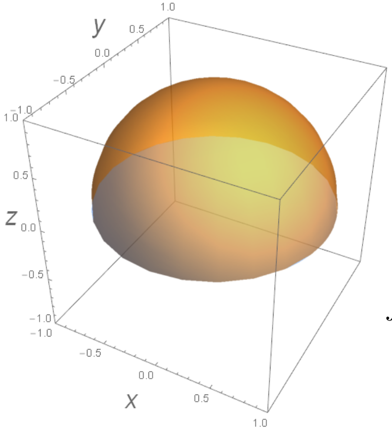
<u>Example</u>: Integrate the function $f(x, y, z) = \frac{z}{\sqrt{x^2+y^2}}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$.



- We use spherical coordinates.
- The sphere is $\rho = 1$, and the upper half corresponds to $0 \leq \varphi \leq \pi/2$.
- There are no restrictions on θ, so the region is $0 \leq \rho \leq 1$, $\pi/2 \leq \varphi \leq \pi$, and $0 \leq \theta \leq 2\pi$.

Cylindrical and Spherical, IV

<u>Example</u>: Integrate the function $f(x, y, z) = \frac{z}{\sqrt{x^2+y^2}}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$.



- The function is $f(\rho, \theta, \varphi) = \dfrac{\rho\cos(\varphi)}{\sqrt{\rho^2\sin^2(\varphi)}} = \dfrac{\cos(\varphi)}{\sin(\varphi)}$ and the differential is $\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$.
- So the integral is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \frac{\cos(\varphi)}{\sin(\varphi)}\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cos(\varphi)\,d\rho\,d\varphi\,d\theta$$

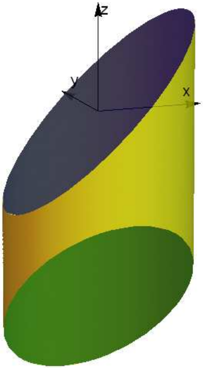
$$= 2\pi/3.$$

Cylindrical and Spherical, V

Example: Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the region inside $x^2 + y^2 = 1$, below the plane $z = x$, and above the plane $z = -2 - y$.

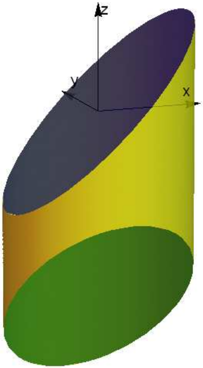
Cylindrical and Spherical, V

Example: Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the region inside $x^2 + y^2 = 1$, below the plane $z = x$, and above the plane $z = -2 - y$.



Cylindrical and Spherical, V

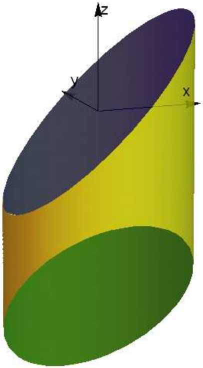
Example: Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the region inside $x^2 + y^2 = 1$, below the plane $z = x$, and above the plane $z = -2 - y$.



- The surface $x^2 + y^2 = 1$ is a cylinder, and the other two bounding curves are functions of z.
- The function also involves $x^2 + y^2$.
- All of these things indicate that we should use cylindrical coordinates.

Cylindrical and Spherical, VI

Example: Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the region inside $x^2 + y^2 = 1$, below the plane $z = x$, and above the plane $z = -2 - y$.



- The integration bounds are $0 \le \theta \le 2\pi$, $0 \le r \le 1$, and $-2 - r\sin(\theta) \le z \le r\cos(\theta)$.
- The function is $\sqrt{x^2 + y^2} = r$, with differential $r\,dz\,dr\,d\theta$.
- Thus, in cylindrical coordinates the integral is

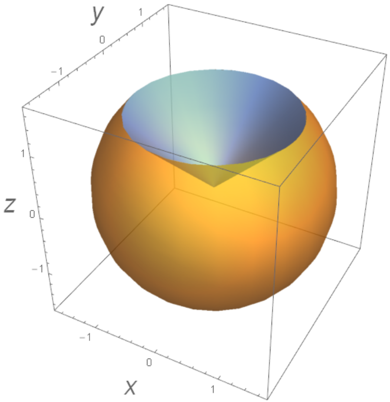
$$\int_0^{2\pi} \int_0^1 \int_{-2-r\sin(\theta)}^{r\cos(\theta)} r \cdot r\,dz\,dr\,d\theta = \frac{4\pi}{3}.$$

Cylindrical and Spherical, VII

Example: Integrate the function $f(x, y, z) = 1$ on the region below $z = \sqrt{x^2 + y^2}$ and inside $x^2 + y^2 + z^2 = 3$.

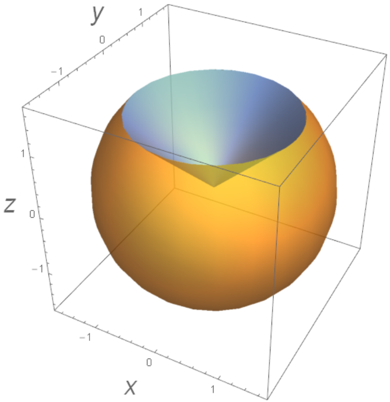
Cylindrical and Spherical, VII

Example: Integrate the function $f(x, y, z) = 1$ on the region below $z = \sqrt{x^2 + y^2}$ and inside $x^2 + y^2 + z^2 = 3$.



Cylindrical and Spherical, VII

Example: Integrate the function $f(x, y, z) = 1$ on the region below $z = \sqrt{x^2 + y^2}$ and inside $x^2 + y^2 + z^2 = 3$.



- We use spherical coordinates.
- The cone becomes $\varphi = \pi/4$, the xy-plane is while the sphere is $\rho = \sqrt{3}$.
- The function is 1 and $dV = \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$.
- Thus, the integral is

$$\int_0^{2\pi} \int_{\pi/4}^{\pi} \int_0^{\sqrt{3}} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

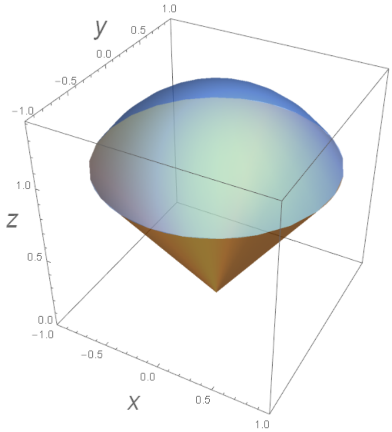
$$= (2\sqrt{3} + \sqrt{6})\pi.$$

Cylindrical and Spherical, VIII

Example: Find $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$

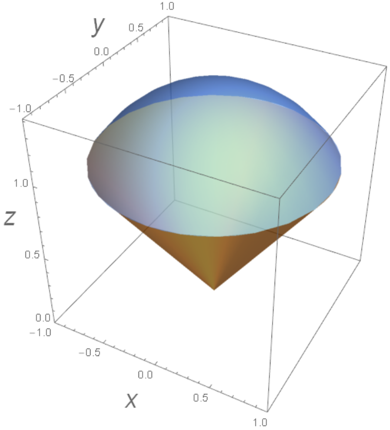
Cylindrical and Spherical, VIII

Example: Find $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$.



Cylindrical and Spherical, VIII

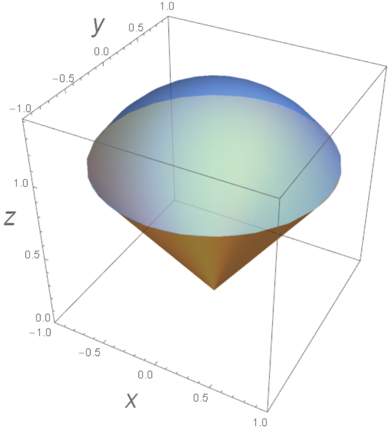
Example: Find $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$.



- This is the integral of $f(x, y, z) = z\sqrt{x^2 + y^2}$ over the solid region D defined by $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, $\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$.
- The region is the solid lying below the hemisphere $z = \sqrt{2 - x^2 - y^2}$ and above the cone $z = \sqrt{x^2 + y^2}$.

Cylindrical and Spherical, IX

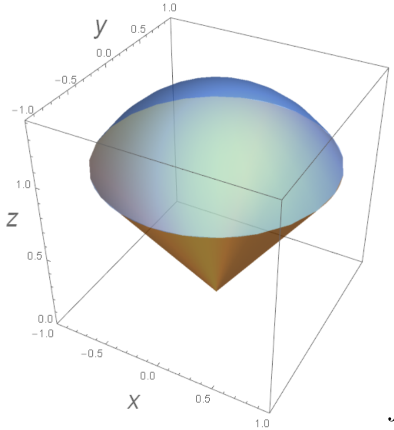
Example: Find $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$.



- Those surfaces have simple descriptions in spherical coordinates, as does the function $f(x, y, z) = \sqrt{x^2+y^2+z^2}$.
- The hemisphere $z = \sqrt{2-x^2-y^2}$ becomes $\rho = \sqrt{2}$, while the cone $z = \sqrt{x^2+y^2}$ becomes $\varphi = \pi/4$.

Cylindrical and Spherical, X

Example: Find $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$.



- We want the region above the cone, so $0 \le \varphi \le \pi/4$. We also have $0 \le \theta \le 2\pi$ and $0 \le \rho \le \sqrt{2}$.
- The function is $\sqrt{x^2+y^2+z^2} = \rho$, and the differential is $dV = \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$.
- The integral is therefore

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho \cdot \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$$
$$= (2\pi)(1-\sqrt{2}/2).$$

Applications of Integration, I

One straightforward but still very useful application of multiple integration is to computing areas of regions in the plane, and volumes of regions in space.

- The central ideas are that

$$\begin{aligned} \text{Area}(R) &= \iint_R 1\, dA \\ \text{Volume}(D) &= \iiint_D 1\, dV. \end{aligned}$$

- Thus, if we can describe a region in a form that lends itself to integration, we can calculate the region's area (or volume).

A closely related problem is to calculate the average value of a function on a region.

- To calculate the average value of f on a region, we integrate f on the region and then divide by the region's area or volume.

Applications of Integration, II

Example: Let R be the plane region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.

1. Find the area of R.
2. Find the average value of y on R.

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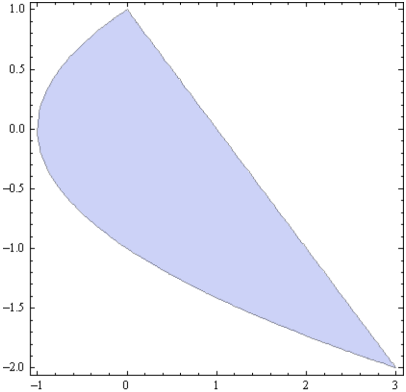

Applications of Integration, II

<u>Example</u>: Let R be the plane region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.

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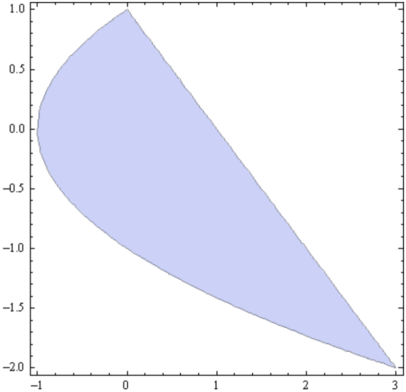

- We can set up the integrals with either integration order $dy\,dx$ or $dx\,dy$.
- But if we use vertical slices, we have to divide the region into two pieces.
- So, we will use horizontal slices with the integration order $dx\,dy$.

Applications of Integration, II

Example: Let R be the plane region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.

1. Find the area of R.

Applications of Integration, II

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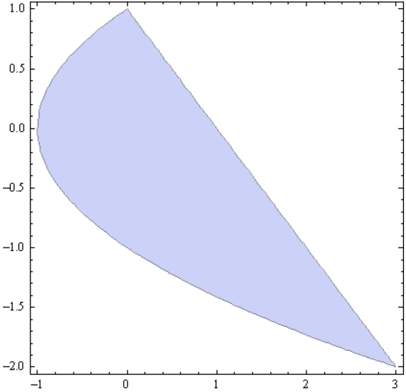

Applications of Integration, II

Example: Let R be the plane region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.

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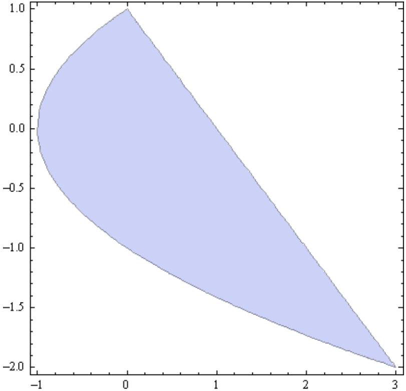

- For the area, the function is $f(x, y) = 1$.
- With order $dx\,dy$, the integral is
$$\int_{-2}^{1} \int_{y^2-1}^{1-y} 1\,dx\,dy = \int_{-2}^{1} x\Big|_{x=y^2-1}^{1-y}\,dy = \int_{-2}^{1} \left[2 - y - y^2\right]\,dy = \frac{9}{2}.$$

Applications of Integration, III

Example: Let R be the plane region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.

2. Find the average value of y on R.

Applications of Integration, III

Example: Let R be the plane region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.

2. Find the average value of y on R.

- For the average value of y, we integrate $f(x, y) = y$ and then divide by the area, which we just calculated as $9/2$.
- With order $dx\,dy$, the integral is $\int_{-2}^{1} \int_{y^2-1}^{1-y} y\,dx\,dy$

  $= \int_{-2}^{1} xy \Big|_{x=y^2-1}^{1-y} dy = \int_{-2}^{1} \left[2y - y^2 - y^3\right] dy = -\frac{9}{4}.$
- Thus, the average value is $(-9/4)/(9/2) = -1/2$.

Applications of Integration, IV

Example: Find the volume of the sphere $x^2 + y^2 + z^2 = 4$.

Applications of Integration, IV

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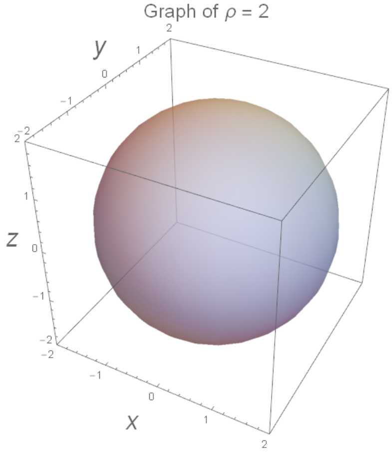
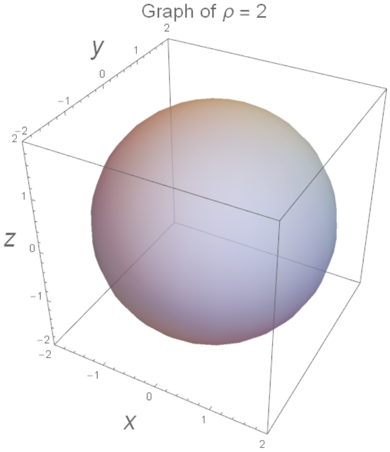


Applications of Integration, IV

Example: Find the volume of the sphere $x^2 + y^2 + z^2 = 4$.



- We want to evaluate $\iiint_D 1\, dV$, where D is interior of the given sphere.
- In spherical coordinates, it is the sphere $\rho = 2$.
- So, the desired integral is $\int_0^{2\pi} \int_0^{\pi} \int_0^{2} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$ $= \int_0^{2\pi} \int_0^{\pi} \frac{8}{3} \sin(\varphi)\, d\varphi\, d\theta$ $= \int_0^{2\pi} \frac{16}{3}\, d\theta = \frac{32\pi}{3}$.

Summary

We did more examples of integrals in cylindrical and spherical coordinates.

We discussed how to use double and triple integrals to compute areas, volumes, and average values.

Next lecture: More applications of integration.