Math 2321 (Multivariable Calculus)

Lecture #20 of 38 $\sim$ March 8, 2021

Triple Integrals in Cylindrical and Spherical Coordinates

- Triple Integrals in Cylindrical Coordinates
- Spherical Coordinates
- Triple Integrals in Spherical Coordinates

This material represents §3.3.4-3.3.5 from the course notes.

Cylindrical Coordinates Reminders, I

We introduced cylindrical coordinates last time:

Definition

The <u>*cylindrical coordinates*</u> (r, θ, z) *of a point whose rectangular coordinates are* (x, y, z) *satisfy* $x = r\cos(\theta)$, $y = r\sin(\theta)$, *and* $z = z$ *for* $r \geq 0$ *and* $0 \leq \theta \leq 2\pi$.

Cylindrical coordinates are a simple three-dimensional version of polar coordinates: we merely include the z-coordinate along with the polar coordinates r and θ.

- To convert from rectangular to cylindrical, we have $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ (possibly plus π depending on the signs of x and y), and obviously $z = z$.

Cylindrical Coordinates Reminders, II

The parameters r and θ are essentially the same as in polar.

- Explicitly, r measures the distance of a point to the z-axis.
- Also, θ measures the angle (in a horizontal plane) from the positive x-direction.

Cylindrical coordinates are useful in simplifying regions that have a circular symmetry.

- In particular, the cylinder $x^2 + y^2 = a^2$ in 3-dimensional rectangular coordinates has the much simpler equation $r = a$ in cylindrical.
- Likewise, the cone $z = a\sqrt{x^2 + y^2}$ has the much simpler equation $z = ar$.
- More generally, $z = f(r)$ is the surface of revolution obtained by revolving the graph of $z = f(x)$ around the z-axis.

Cylindrical Coordinates Reminders, III

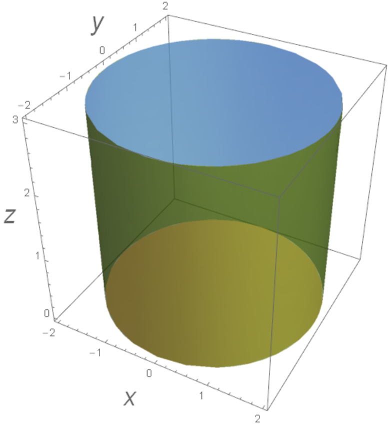
Like most of our other triple integrals, the most difficult part is setting up the integral.

- When we want to set up a triple integral in cylindrical coordinates with integration order $dz\, dr\, d\theta$, we can project the solid into the xy-plane (equivalently, the $r\theta$-plane) and then set up the r and θ limits just as in polar coordinates.
- We can then find the z limits just as with triple integrals in rectangular coordinates: the lower z limit is the equation of the lower bounding surface, while the upper z limit is the equation of the upper bounding surface.
- The volume differential in cylindrical coordinates is

$$\boxed{dV = r\, dz\, dr\, d\theta}.$$

Integration in Cylindrical Coordinates, I

Example: Set up and evaluate $\iiint_D \sqrt{x^2+y^2}\,dV$ where D is the region with $0 \le z \le 3$ inside the cylinder $x^2+y^2=4$.



Integration in Cylindrical Coordinates, I

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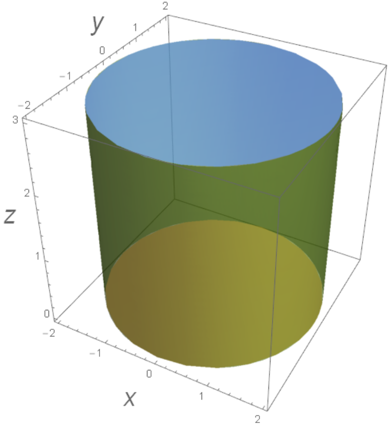


- Since $\sqrt{x^2 + y^2} = r$, the function is simply $f(r, \theta, z) = r$, and the cylindrical differential is $r\, dz\, dr\, d\theta$.
- The integral is therefore

$$\int_0^{2\pi} \int_0^2 \int_0^3 r \cdot r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 3r^2 dr\, d\theta$$

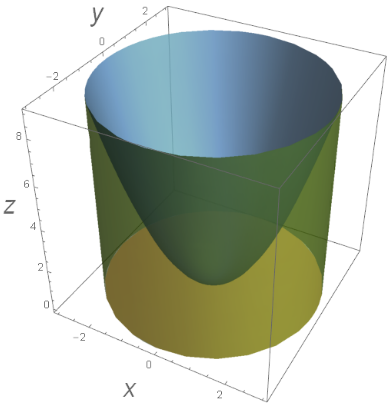
$$= \int_0^{2\pi} 8\, d\theta = 16\pi.$$

Integration in Cylindrical Coordinates, II

Example: Set up and evaluate $\iiint_D z\,dV$ where D is the region inside $x^2 + y^2 = 9$, below $z = x^2 + y^2$, and above $z = 0$.

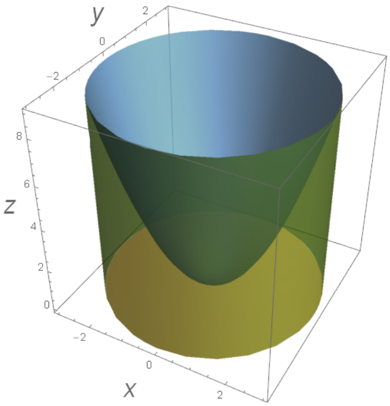
Integration in Cylindrical Coordinates, II

<u>Example</u>: Set up and evaluate $\iiint_D z\,dV$ where D is the region inside $x^2 + y^2 = 9$, below $z = x^2 + y^2$, and above $z = 0$.



Integration in Cylindrical Coordinates, II

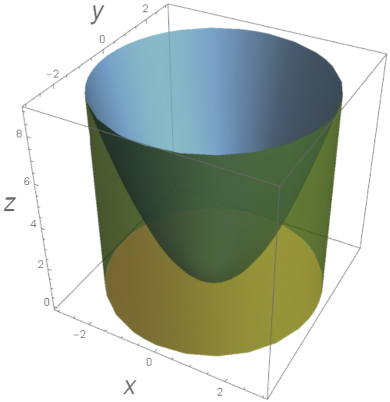
<u>Example</u>: Set up and evaluate $\iiint_D z\,dV$ where D is the region inside $x^2 + y^2 = 9$, below $z = x^2 + y^2$, and above $z = 0$.



- We use cylindrical coordinates, since the bounding surfaces are $r = 3$, $z = r^2$, and $z = 0$ in cylindrical.
- There are no restrictions on θ, so we have $0 \leq \theta \leq 2\pi$. Also, we have $0 \leq r \leq 3$, and then $0 \leq z \leq r^2$.
- The function is simply $f(r, \theta, z) = z$, and the differential is $r\,dz\,dr\,d\theta$.

Integration in Cylindrical Coordinates, III

Example: Set up and evaluate $\iiint_D z\,dV$ where D is the region inside $x^2+y^2=9$, below $z=x^2+y^2$, and above $z=0$.



- The integral is therefore

$$\int_0^{2\pi}\int_0^3\int_0^{r^2} z\cdot r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 \frac{1}{2}rz^2\Big|_{z=0}^{z=r^2}dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3 \frac{1}{2}r^5 dr\,d\theta$$

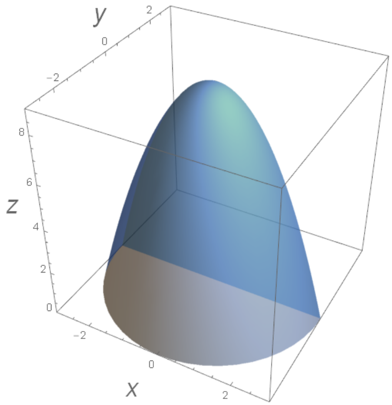
$$=\int_0^{2\pi}\frac{243}{4}\,d\theta=\frac{243\pi}{2}.$$

Integration in Cylindrical Coordinates, IV

Example: Integrate the function $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$ on the region underneath $z = 9 - x^2 - y^2$, above the xy-plane, with $y \leq 0$.

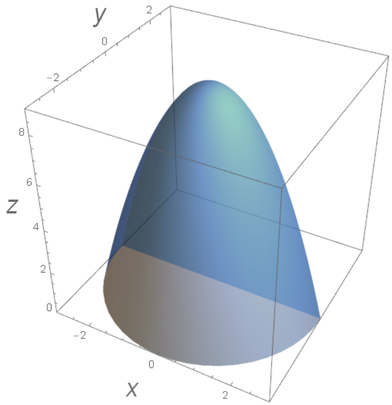
Integration in Cylindrical Coordinates, IV

<u>Example</u>: Integrate the function $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$ on the region underneath $z = 9 - x^2 - y^2$, above the xy-plane, with $y \leq 0$.



Integration in Cylindrical Coordinates, IV

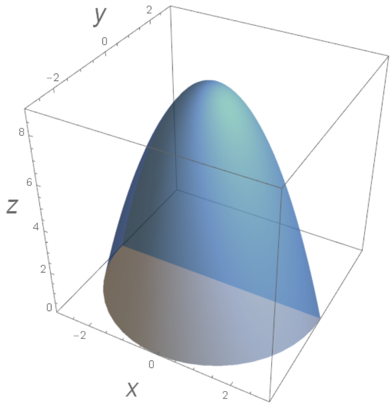
Example: Integrate the function $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$ on the region underneath $z = 9 - x^2 - y^2$, above the xy-plane, with $y \leq 0$.



- We set up in cylindrical: the paraboloid has equation $z = 9 - r^2$, so the part with $z \geq 0$ has $0 \leq r \leq 3$.
- Here, we have $\pi \leq \theta \leq 2\pi$, and also $0 \leq z \leq 9 - r^2$.
- Since $\sqrt{x^2 + y^2} = r$, the function is simply $f = 1/r$, and the differential is $r\,dz\,dr\,d\theta$.

Integration in Cylindrical Coordinates, V

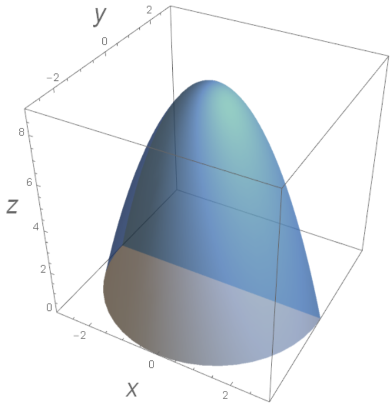
Example: Integrate the function $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$ on the region underneath $z = 9 - x^2 - y^2$, above the xy-plane, with $y \leq 0$.



- The desired integral is
$$\int_{\pi}^{2\pi} \int_0^3 \int_0^{9-r^2} \frac{1}{r} \cdot r \, dz \, dr \, d\theta$$

Integration in Cylindrical Coordinates, V

Example: Integrate the function $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$ on the region underneath $z = 9 - x^2 - y^2$, above the xy-plane, with $y \leq 0$.



- The desired integral is

$$\int_{\pi}^{2\pi} \int_0^3 \int_0^{9-r^2} \frac{1}{r} \cdot r \, dz \, dr \, d\theta$$
$$= \int_{\pi}^{2\pi} \int_0^3 \int_0^{9-r^2} 1 \, dz \, dr \, d\theta$$
$$= \int_{\pi}^{2\pi} \int_0^3 (9 - r^2) \, dr \, d\theta$$
$$= \int_{\pi}^{2\pi} (9r - \frac{1}{3} r^3) \Big|_{r=0}^{3} \, d\theta$$
$$= \int_{\pi}^{2\pi} 18 \, d\theta = 18\pi.$$

Integration in Cylindrical Coordinates, VI

Example: Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

Integration in Cylindrical Coordinates, VI

Example: Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

- This is an iterated integral of the function $f(x,y,z) = z\sqrt{x^2+y^2}$ over the solid region D defined by the inequalities $0 \le x \le 2$, $0 \le y \le \sqrt{4-x^2}$, $\sqrt{x^2+y^2} \le z \le 2$.
- Notice that the x and y limits describe the region $0 \le x \le 2$, $0 \le y \le \sqrt{4-x^2}$, which is a quarter-disc.
- This, along with the presence of $\sqrt{x^2+y^2}$ in the z-limit and in the function, strongly suggest converting to cylindrical coordinates.

Integration in Cylindrical Coordinates, VII

Example: Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 z\sqrt{x^2+y^2}\, dz\, dy\, dx$.

Integration in Cylindrical Coordinates, VII

Example: Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 z\sqrt{x^2+y^2}\,dz\,dy\,dx$.

- In cylindrical coordinates, we can see that the xy-region becomes $0 \le r \le 2$, $0 \le \theta \le \pi/2$. Also, the range for z becomes $r \le z \le 2$.
- Since $\sqrt{x^2+y^2} = r$, the function is simply $f = zr$, and the cylindrical differential is $r\,dz\,dr\,d\theta$.
- The integral is therefore equal to $\int_0^{\pi/2} \int_0^2 \int_r^2 zr \cdot r\,dz\,dr\,d\theta$.

Integration in Cylindrical Coordinates, VIII

Example: Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 z\sqrt{x^2+y^2}\, dz\, dy\, dx.$

- Now we can evaluate it:

$$\begin{aligned}
\int_0^{\pi/2} \int_0^2 \int_r^2 zr \cdot r \, dz\, dr\, d\theta &= \int_0^{\pi/2} \int_0^2 \left[\frac{1}{2} z^2 r^2\right]\Big|_{z=r}^2 dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^2 \left[2r^2 - \frac{1}{2} r^4\right] dr\, d\theta \\
&= \int_0^{\pi/2} \left[\frac{2}{3} r^3 - \frac{1}{10} r^5\right]\Big|_{r=0}^2 d\theta \\
&= \int_0^{\pi/2} \left(\frac{16}{3} - \frac{32}{10}\right) d\theta \\
&= \int_0^{\pi/2} \frac{32}{15}\, d\theta = \frac{16\pi}{15}.
\end{aligned}$$

Spherical Coordinates, I

Cylindrical coordinates are very useful for evaluating integrals with circular symmetries.

- However, we often want to integrate over spherical regions too.
- The sphere $x^2 + y^2 + z^2 = 1$ does not have such a nice description in cylindrical: it is $r^2 + z^2 = 1$, which requires taking square roots when we set up the z-limits.

For this reason, we also have another 3-dimensional coordinate system, spherical coordinates, which we use for simplifying integrals involving spheres.

- If you like, take a moment to imagine that you are located on a sphere, and consider how you could describe your position on the sphere to someone else.

Spherical Coordinates, II

Spherical coordinates are defined as follows:

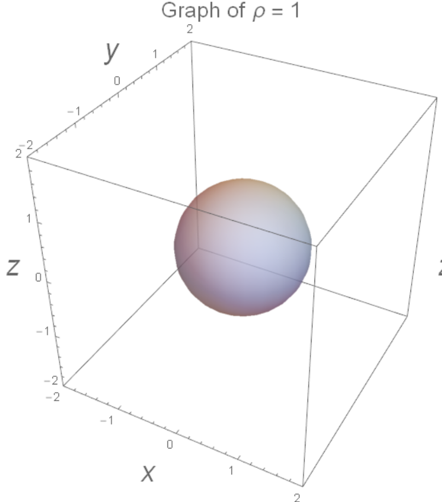
Definition

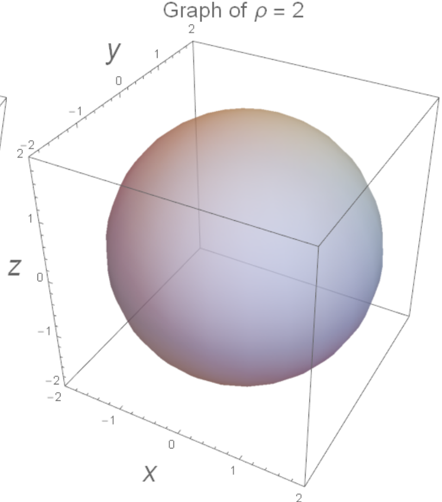
The spherical coordinates (ρ, θ, φ) *of a point* (x, y, z) *satisfy* $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$ *for* $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, *and* $0 \leq \varphi \leq \pi$.

- The parameters θ and φ are angles: θ measures longitude, while φ measures latitude.
- The parameter ρ measures the distance to the origin.
- To find the spherical coordinates of a point in (x, y, z) in rectangular coordinates, we have $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$ and $\varphi = \tan^{-1}(r/z) = \tan^{-1}(\sqrt{x^2 + y^2}/z)$, while θ has the same definition as in cylindrical coordinates.

Spherical Coordinates, III

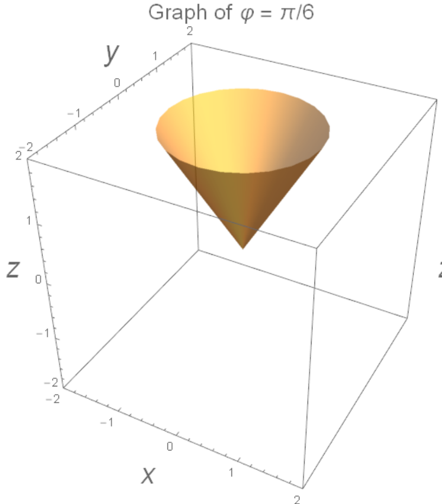
The parameter ρ measures the distance from the origin $(0,0,0)$, and so the equation $\rho = c$ is the sphere $x^2 + y^2 + z^2 = c^2$:

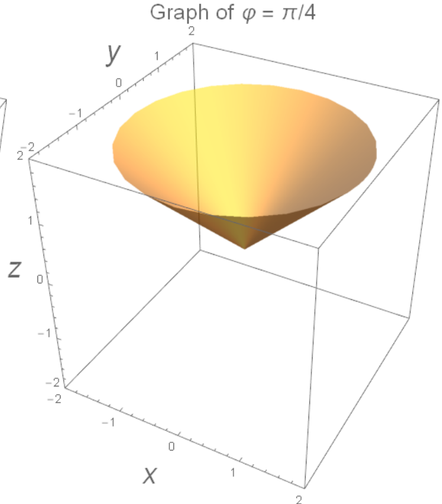




Spherical Coordinates, IV

The parameter φ measures the angle downward from the positive z-axis, so $\varphi = c$ is the cone $z = \tan(\varphi) r$:

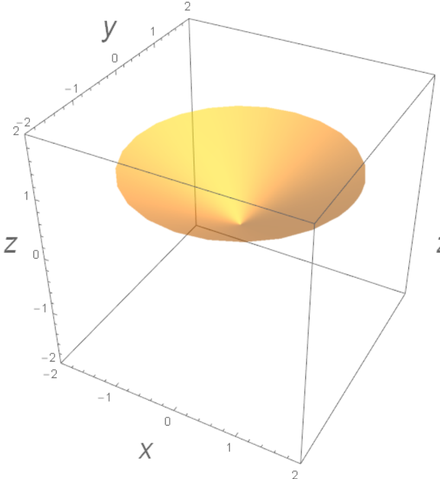




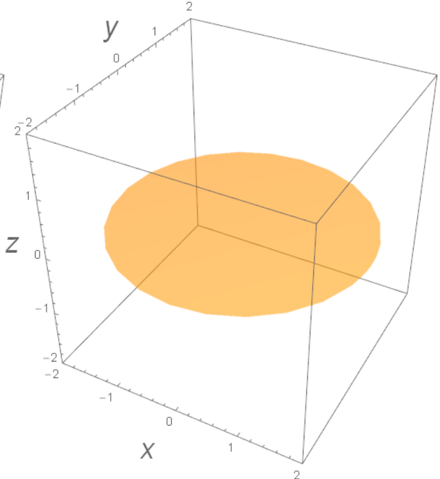
Spherical Coordinates, V

The parameter φ measures the angle downward from the positive z-axis:

Graph of $\varphi = \pi/3$

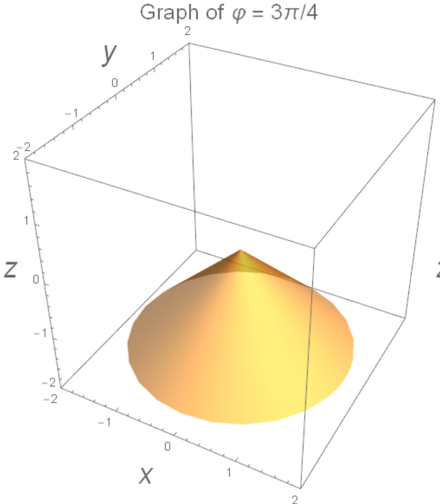


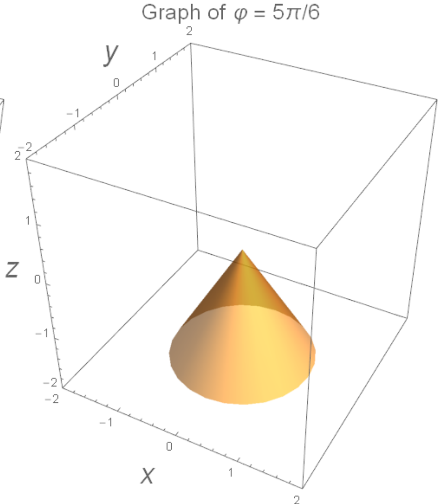
Graph of $\varphi = \pi/2$



Spherical Coordinates, VI

The parameter φ measures the angle downward from the positive z-axis:





Spherical Coordinates, VII

Example: Perform the following coordinate conversions:

1. Find rectangular coordinates for $(\rho, \theta, \varphi) = (4, 3\pi/2, \pi/2)$.
2. Find spherical coordinates for $(x, y, z) = (1, 1, \sqrt{2})$.
3. Find rectangular coordinates for $(\rho, \theta, \varphi) = (8, \pi/6, \pi/3)$.
4. Find spherical coordinates for $(r, \theta, z) = (2, \pi, -2)$.
5. Find cylindrical coordinates for $(\rho, \theta, \varphi) = (8, \pi/2, \pi/3)$.

Spherical Coordinates, VII

Example: Perform the following coordinate conversions:

1. Find rectangular coordinates for $(\rho, \theta, \varphi) = (4, 3\pi/2, \pi/2)$.
2. Find spherical coordinates for $(x, y, z) = (1, 1, \sqrt{2})$.
3. Find rectangular coordinates for $(\rho, \theta, \varphi) = (8, \pi/6, \pi/3)$.
4. Find spherical coordinates for $(r, \theta, z) = (2, \pi, -2)$.
5. Find cylindrical coordinates for $(\rho, \theta, \varphi) = (8, \pi/2, \pi/3)$.

- For $(\rho, \theta, \varphi) = (4, 3\pi/2, \pi/2)$ we have $(x, y, z) = (0, -4, 0)$.
- For $(x, y, z) = (1, 1, \sqrt{2})$ we have $(\rho, \theta, \varphi) = (2, \pi/4, \pi/4)$.
- For $(\rho, \theta, \varphi) = (8, \pi/6, \pi/3)$ we have $(x, y, z) = (6, 2\sqrt{3}, 4)$.
- For $(r, \theta, z) = (2, \pi, -2)$ we get $(\rho, \theta, \varphi) = (2\sqrt{2}, \pi, 3\pi/4)$.
- For $(\rho, \theta, \varphi) = (8, \pi/2, \pi/3)$ we get $(r, \theta, z) = (4\sqrt{3}, \pi/2, 4)$.

Spherical Coordinates, VIII

Spherical coordinates are most useful when integrating over regions with spherical symmetries. (Not so surprising, given the name....)

- In spherical, the sphere $x^2 + y^2 + z^2 = a^2$ has the much simpler equation $\rho = a$.
- Also, the cone $az = \sqrt{x^2 + y^2}$ is quite simple: $\varphi = \tan^{-1}(a)$.
- Some common examples: $z = \sqrt{3(x^2 + y^2)}$ is $\varphi = \pi/6$, $z = \sqrt{x^2 + y^2}$ is $\varphi = \pi/4$, $z = 0$ is $\varphi = \pi/2$, $z = -\sqrt{x^2 + y^2}$ is $\varphi = 3\pi/4$, etc.
- We typically set up spherical integrals with the integration order $d\rho\, d\varphi\, d\theta$, because typically the ρ bounds are the most complicated, while the θ bounds are the simplest.

Spherical Coordinates, IX

It remains to compute the spherical volume differential dV.

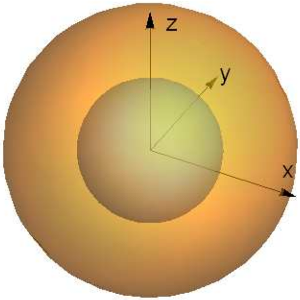
- With $x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta,\ z = \rho \cos\varphi$, we get
$$J = \begin{vmatrix} \cos\theta \sin\varphi & \rho \cos\theta \cos\varphi & -\rho \sin\theta \sin\varphi \\ \sin\theta \sin\varphi & \rho \sin\theta \cos\varphi & \rho \cos\theta \sin\varphi \\ \cos\varphi & -\rho \sin\varphi & 0 \end{vmatrix} = \rho^2 \sin\varphi.$$
- Thus the differential in spherical coordinates is
$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}.$$
- This one is not quite as easy to remember as the cylindrical area differential. It must simply be memorized.

Spherical Coordinates, X

Example: Find $\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV$ where D is the region $1 \le x^2 + y^2 + z^2 \le 4$.

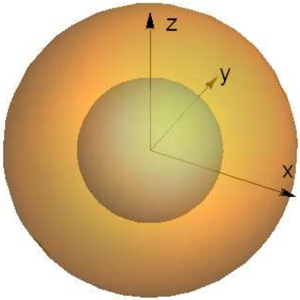
Spherical Coordinates, X

<u>Example</u>: Find $\iiint_D \sqrt{x^2+y^2+z^2}\, dV$ where D is the region $1 \le x^2+y^2+z^2 \le 4$.



Spherical Coordinates, X

Example: Find $\iiint_D \sqrt{x^2+y^2+z^2}\, dV$ where D is the region $1 \leq x^2+y^2+z^2 \leq 4$.



- The region is bounded by the two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$, so we set up in spherical coordinates.
- The first sphere is $\rho = 1$ and the second is $\rho = 2$.
- There are no restrictions on φ and θ.
- Thus, the region of integration is $1 \leq \rho \leq 2$, $0 \leq \varphi \leq \pi$, and $0 \leq \theta \leq 2\pi$.

Spherical Coordinates, XI

Example: Find $\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV$ where D is the region $1 \le x^2 + y^2 + z^2 \le 4$.

Spherical Coordinates, XI

Example: Find $\iiint_D \sqrt{x^2+y^2+z^2}\,dV$ where D is the region $1 \le x^2+y^2+z^2 \le 4$.

- The region is $1 \le \rho \le 2$, $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$.
- The function is $\sqrt{x^2+y^2+z^2} = \rho$ and the differential is $\rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$.
- The integral in spherical coordinates is therefore $\int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$

Spherical Coordinates, XI

Example: Find $\iiint_D \sqrt{x^2+y^2+z^2}\,dV$ where D is the region $1 \le x^2+y^2+z^2 \le 4$.

- The region is $1 \le \rho \le 2$, $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$.
- The function is $\sqrt{x^2+y^2+z^2} = \rho$ and the differential is $\rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$.
- The integral in spherical coordinates is therefore

$$\int_0^{2\pi}\int_0^{\pi}\int_1^{2} \rho \cdot \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \rho^3 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\rho^4 \sin(\varphi)\Big|_{\rho=1}^{2}\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{15}{4}\sin(\varphi)\,d\varphi\,d\theta = \int_0^{2\pi}\frac{15}{2}\,d\theta = 15\pi.$$

Summary

We discussed how to set up triple integrals in cylindrical coordinates.

We introduced spherical coordinates and how to set up triple integrals in spherical coordinates.

Next lecture: More triple integrals in cylindrical and spherical coordinates, applications of integration.