Math 2321 (Multivariable Calculus)

Lecture #19 of 38 $\sim$ March 4, 2021

Triple Integrals + Change of Coordinates

- More Triple Integrals
- General Changes of Coordinates
- Cylindrical Coordinates

This material represents §3.3.1 + 3.3.4-3.3.5 from the course notes.

More Triple Integrals, I

Example: Set up an iterated integral for each of the following:

4. $\iiint_D z\, dV$ where D is the region bounded by $y + z = 1$, $y = x^2$, and the xy-plane.

More Triple Integrals, I

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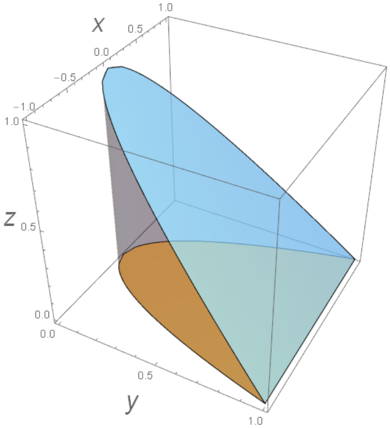

- For $dz\,dy\,dx$, we project into the xy-plane.

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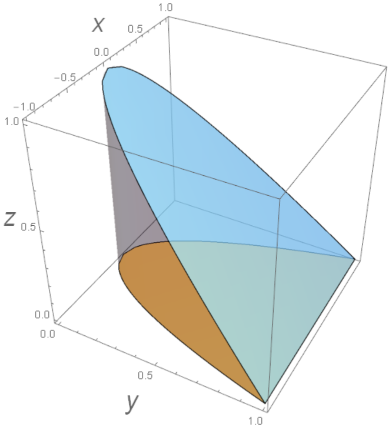

- For $dz\,dy\,dx$, we project into the xy-plane.
- The region lies between $y = 1$ and $y = x^2$.
- So, we have $-1 \le x \le 1$ and $x^2 \le y \le 1$.
- The z limits are $z = 0$ and $z = 1 - y$.
- Thus, our triple integral is
$$\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} z\,dz\,dy\,dx.$$

More Triple Integrals, II

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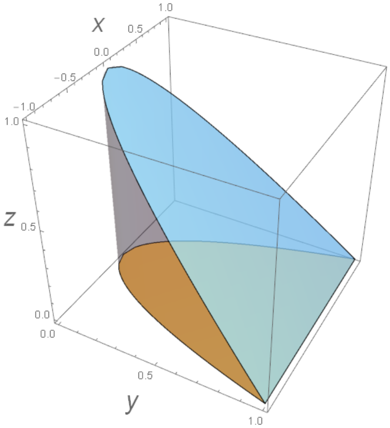

- We could also do $dx\,dz\,dy$ by projecting into the yz-plane.

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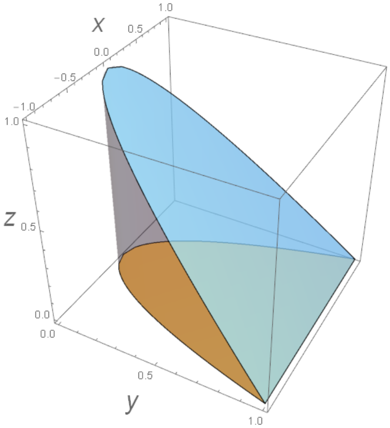

- We could also do $dx\,dz\,dy$ by projecting into the yz-plane.
- The region is $0 \le y \le 1$, $0 \le z \le 1-y$.
- The x limits are $-\sqrt{y}$ (front) and $\sqrt{y}$ (back).
- Thus, our triple integral is $$\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} z\,dx\,dz\,dy.$$

More Triple Integrals, III

Example: Set up an iterated integral for each of the following:

5. The integral of $f(x, y, z) = x$ on the region with $x, y, z \geq 0$, below $x + z = 1$, and also below $y^2 + z = 1$.

More Triple Integrals, III

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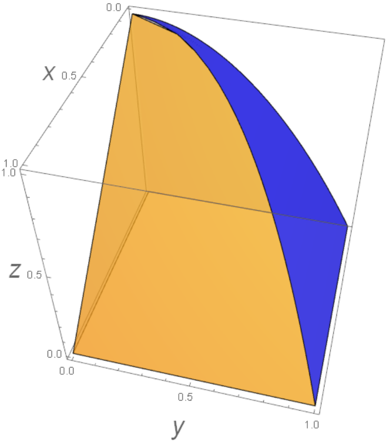

- If we use $dz\,dy\,dx$ and project into the xy-plane, we will have to divide into two regions, because the top surface changes in the middle of the region.
- It is better to use a different integration order here, where we project into the xz or yz plane.

More Triple Integrals, IV

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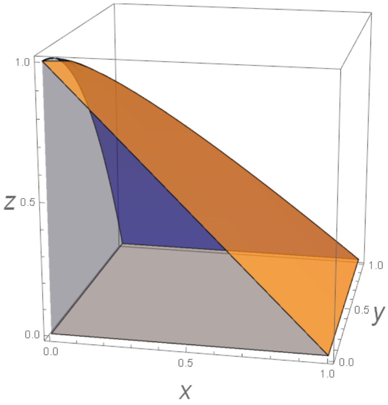

- For $dy\,dz\,dx$, we project into the xz-plane.

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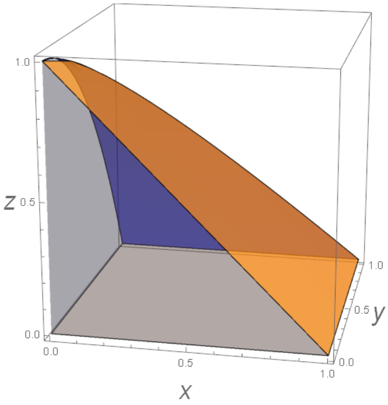

- For $dy\,dz\,dx$, we project into the xz-plane.
- The region lies between $z = 0$ and $z = 1 - x$.
- So, we have $0 \leq x \leq 1$ and $0 \leq z \leq 1 - x$.
- Then y ranges from 0 (front) to $\sqrt{1 - z}$ (back).
- Thus, our triple integral is
$$\int_0^1 \int_0^{1-x} \int_0^{\sqrt{1-z}} x \, dy \, dz \, dx.$$

More Triple Integrals, V

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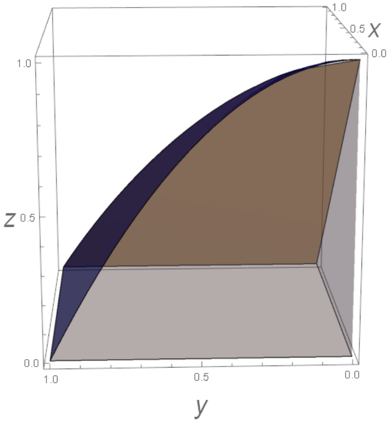

- For $dx\,dz\,dy$, we project into the yz-plane.

More Triple Integrals, V

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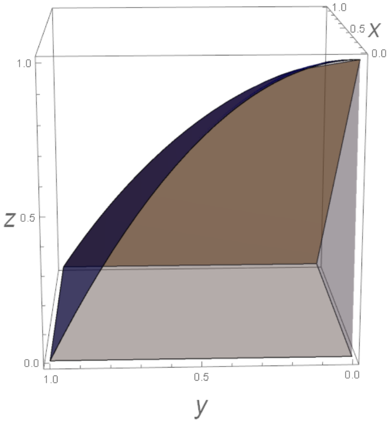

- For $dx\, dz\, dy$, we project into the yz-plane.
- The region lies between $z = 0$ and $z = 1 - y^2$.
- So, we have $0 \leq y \leq 1$ and $0 \leq z \leq 1 - y^2$.
- Then x ranges from 0 (front) to $1 - z$ (back).
- Thus, our triple integral is
$$\int_0^1 \int_0^{1-y^2} \int_0^{1-z} x\, dx\, dz\, dy.$$

More Triple Integrals, VI

Example: Set up an iterated integral for each of the following:

6. $\iiint_D 1\,dV$ where D is the region cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = 0$ and $z = x + 2$.

More Triple Integrals, VI

Example: Set up an iterated integral for each of the following:

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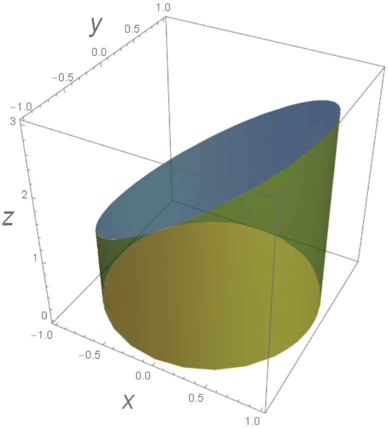

- We try $dz\,dy\,dx$.

More Triple Integrals, VI

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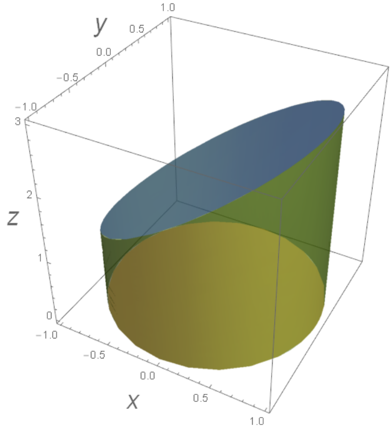


- We try $dz\,dy\,dx$.
- The xy-region is the interior of the circle $x^2 + y^2 = 1$, which we can describe as $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$.
- Then z ranges from 0 (bottom) to $x + 2$ (top).
- Thus, our triple integral is $$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{x+2} 1\,dz\,dy\,dx.$$

Change of Coordinates, I

So far, we have primarily discussed double and triple integrals in rectangular coordinates.

- We will now talk about the multivariable equivalent of the one-variable integration technique of substitution.
- If we think about substitution as being a "change of variables" to a new system of coordinates, then the answer is yes: we can rewrite multiple integrals in different coordinate systems.
- In fact, we have already discussed an alternative coordinate system: namely, polar coordinates.
- Our goal today is to discuss how integrals transform under general changes of coordinates.

Change of Coordinates, II

Let's review how we transform integrals from rectangular to polar, as a prototype for other coordinate changes.

- If we want to set up $\iint_R f(x, y)\, dA$, there are three things we need to convert into polar coordinates:
  1. The region of integration.
  2. The function $f(x, y)$.
  3. The differential dA.
- The same situation will hold if we want to transform into a different coordinate system.

Change of Coordinates, III

For a change of coordinates in a double integral, we will have two new variables: let's call them s and t.

- Specifically, suppose that we write $x = x(s,t)$ and $y = y(s,t)$ in terms of s and t.
- Then it is very easy to convert the function $f(x,y)$ into a function of s and t: we simply plug in the expressions for x and y in terms of s and f.
- We can also transform the region into the new st-coordinates, much like we did with polar coordinates.
- The only other question is: how do we convert the area differential dA?

Change of Coordinates, IV

So, consider $dA = dy\,dx$.

- If we change s slightly, then x and y will both change: specifically, the change is $\Delta\mathbf{v} \approx \langle x_s \Delta s, y_s \Delta s \rangle$.
- Likewise, if we change t slightly, then we get another vector $\Delta\mathbf{w} \approx \langle x_t \Delta t, y_t \Delta t \rangle$.
- These two vectors form a parallelogram, and the area of this parallelogram is $\Delta y\,\Delta x$.
- But the area is also the magnitude of the cross product $||\Delta\mathbf{v} \times \Delta\mathbf{w}|| = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} \Delta s\,\Delta t$.
- Therefore, by taking limits as Δs, Δt approach 0, the area differential is $dA = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} ds\,dt$.

Change of Coordinates, V

We give this "differential change of coordinates" quantity a name:

Definition

Suppose that $x = x(s,t)$ and $y = y(s,t)$ are functions of s and t. Then we define the Jacobian as the determinant
$$J = \frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix}.$$

The idea is that the Jacobian tells us how to convert the area differential into *st*-coordinates: specifically, we have
$$dA = dy\,dx = \frac{\partial(x,y)}{\partial(s,t)}\,dt\,ds.$$

Change of Coordinates, VI

Example: Find the Jacobian for the change of coordinates from rectangular to polar, with $x = r\cos\theta$ and $y = r\sin\theta$.

Change of Coordinates, VI

Example: Find the Jacobian for the change of coordinates from rectangular to polar, with $x = r\cos\theta$ and $y = r\sin\theta$.

- We just have to compute $J = \dfrac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$.
- We get $J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta) = r$.
- Thus, as indeed we saw last week, the polar area differential is $dA = r\,dr\,d\theta$.

Change of Coordinates, VII

Here is the general theorem on changing coordinates in a double integral:

Theorem (General Substitution, 2 variables)

If $f(x, y)$ is continuous on R, and $x = x(s, t)$ and $y = y(s, t)$ are functions of s and t, then

$$\iint_R f(x, y)\, dy\, dx = \iint_{R'} f(x(s, y), y(s, t)) \left| \frac{\partial(x, y)}{\partial(s, t)} \right| dt\, ds$$

where R' is the region R expressed in st-coordinates and $\dfrac{\partial(x, y)}{\partial(s, t)} = J(x, y) = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix}$ is the Jacobian of the coordinate transformation.

Change of Coordinates, VIII

As a warning, for this theorem to apply, the change of coordinates needs to be injective (one-to-one) on the original region of integration.

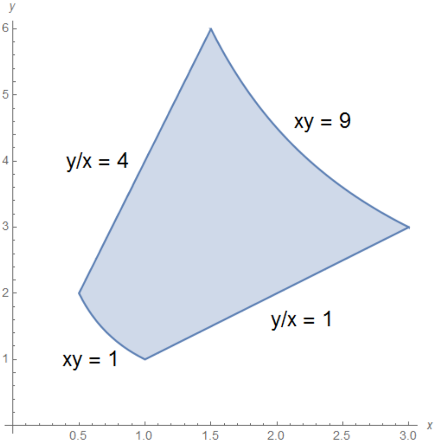
- Specifically, if the image of the old region in the new set of coordinates ranges over some parts of the new region of integration more than once, the formula will be incorrect.
- For example, the change of coordinates $s = \sin(x)$, $t = \sin(y)$ fails this criterion if applied to the square $0 \le x \le 4\pi$, $0 \le y \le 4\pi$, then every point in the image square $-1 \le s \le 1$, $-1 \le t \le 1$ will be covered four times.

Change of Coordinates, IX

Example: Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.

Change of Coordinates, IX

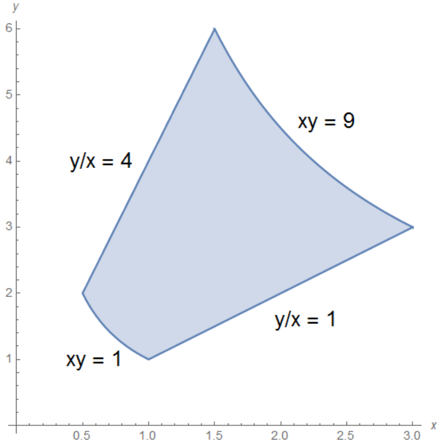
Example: Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.



- We could divide this region into 3 pieces (horizontal or vertical slices).
- But this would be very laborious: we'd have to find all the intersection points, and then set up and evaluate 3 separate double integrals.
- We can save a lot of effort by instead doing a change of variables.

Change of Coordinates, X

Example: Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.



- The inequalities for the region suggest trying $s = xy$ and $t = y/x$, since then the region is just $1 \le s \le 9$, $1 \le t \le 4$.
- Solving for x and y gives $y = \sqrt{st}$, $x = \sqrt{s/t}$, which is one-to-one on the region we have here.

Change of Coordinates, XI

Example: Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.

Change of Coordinates, XI

Example: Evaluate $\iint_R (x^2+y^2)\,dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.

- With $s = xy$ and $t = y/x$, so that $y = \sqrt{st}$, $x = \sqrt{s/t}$, we now transform the region, function, and differential.
- The bounds of integration are $1 \le s \le 9$ and $1 \le t \le 4$.
- The function is $x^2 + y^2 = \dfrac{s}{t} + st$.
- For the new differential, we compute the Jacobian:

$$J = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} \frac{1}{2}s^{-1/2}t^{-1/2} & -\frac{1}{2}s^{1/2}t^{-3/2} \\ \frac{1}{2}s^{-1/2}t^{1/2} & \frac{1}{2}s^{1/2}t^{-1/2} \end{vmatrix}$$

$$= \left(\tfrac{1}{2}s^{-1/2}t^{-1/2}\right)\left(\tfrac{1}{2}s^{1/2}t^{-1/2}\right) - \left(\tfrac{1}{2}s^{-1/2}t^{1/2}\right)\left(-\tfrac{1}{2}s^{1/2}t^{-3/2}\right)$$

$$= \frac{1}{4t} - \left(-\frac{1}{4t}\right) = \frac{1}{2t}.$$ Thus, $dA = \dfrac{1}{2t}\,dt\,ds$.

Change of Coordinates, XII

Example: Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.

- Putting all of this together shows that the integral in st-coordinates is $\displaystyle\int_1^9 \int_1^4 \left(\frac{s}{t} + st\right) \cdot \left(\frac{1}{2t}\right) dt\, ds$.

Change of Coordinates, XII

Example: Evaluate $\iint_R (x^2+y^2)\,dA$ where R is the region in the first quadrant defined by the inequalities $1 \le xy \le 9$, $1 \le y/x \le 4$.

- Putting all of this together shows that the integral in st-coordinates is $\int_1^9 \int_1^4 \left(\frac{s}{t}+st\right)\cdot\left(\frac{1}{2t}\right)\,dt\,ds$.
- Now we can evaluate it:

$$\begin{aligned}
\int_1^9 \int_1^4 \left(\frac{s}{t}+st\right)\left(\frac{1}{2t}\right)\,dt\,ds &= \int_1^9 \int_1^4 \frac{s}{2}\left[t^{-2}+1\right]\,dt\,ds \\
&= \int_1^9 \frac{s}{2}\left[-t^{-1}+t\right]\Big|_{t=1}^{4}\,ds \\
&= \int_1^9 \frac{s}{2}\left[\frac{3}{4}+3\right]ds \\
&= \int_1^9 \frac{15}{8}s\,ds = 75.
\end{aligned}$$

Change of Coordinates, XIII

We can also do changes of coordinates in triple integrals.

- The general procedure is essentially identical, except for having three variables s, t, and u instead of just two: we must transform the region, the function, and the differential.
- The region and function work just as before. The differential is again given by a Jacobian, which is now a 3×3 determinant instead of a 2×2 determinant.
- This 3×3 determinant measures the volume of a 3-dimensional "parallelepiped", just as the 2×2 determinant measured the volume of a 2-dimensional parallelogram.

Change of Coordinates, XIV

Here is the general coordinate-change theorem for triple integrals:

Theorem (General Substitution, 3 variables)

If $f(x, y, z)$ is continuous on D, and $x = x(s, t, u)$, $y = y(s, t, u)$, and $z = z(s, t, u)$ are functions of s, t, u, then

$$\iiint_D f(x, y, z)\, dz\, dy\, dx$$
$$= \iiint_{D'} f(x(s, t, u), y(s, t, u), z(s, t, u)) \left| \frac{\partial(x, y, z)}{\partial(s, t, u)} \right| du\, dt\, ds$$

where D' is the region D expressed in stu-coordinates and

$$J = \frac{\partial(x, y, z)}{\partial(s, t, u)} = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t & \partial x/\partial u \\ \partial y/\partial s & \partial y/\partial t & \partial y/\partial u \\ \partial z/\partial s & \partial z/\partial t & \partial z/\partial u \end{vmatrix}$$ *is the Jacobian of the coordinate transformation.*

Change of Coordinates, XV

In general, unless the region has some kind of obvious description, doing three-dimensional coordinate changes can be very tricky.

- Therefore, we won't bother doing any of these general 3-dimensional coordinate changes.
- We will instead focus on two important 3-dimensional generalizations of polar coordinates that are very useful for simplifying integrals on regions that have circular or spherical symmetries.
- Such situations arise often in physics, chemistry, and engineering due to the spherical symmetry of gravitational and electrical fields.
- These coordinate systems are cylindrical coordinates (r, θ, z) and spherical coordinates (ρ, θ, φ).

Cylindrical Coordinates, I

We start with cylindrical coordinates.

Definition

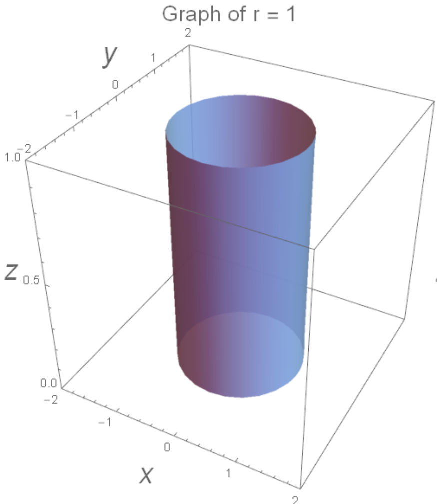
The cylindrical coordinates (r, θ, z) *of a point whose rectangular coordinates are* (x, y, z) *satisfy* $x = r\cos(\theta)$, $y = r\sin(\theta)$, *and* $z = z$ *for* $r \geq 0$ *and* $0 \leq \theta \leq 2\pi$.

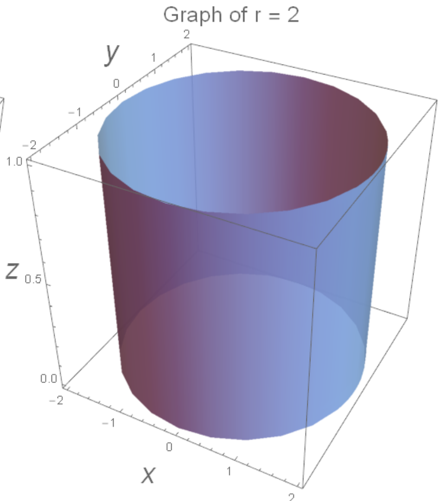
Cylindrical coordinates are a simple three-dimensional version of polar coordinates: we merely include the z-coordinate along with the polar coordinates r and θ.

- To convert from rectangular to cylindrical, we have $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ (possibly plus π depending on the signs of x and y), and obviously $z = z$.

Cylindrical Coordinates, II

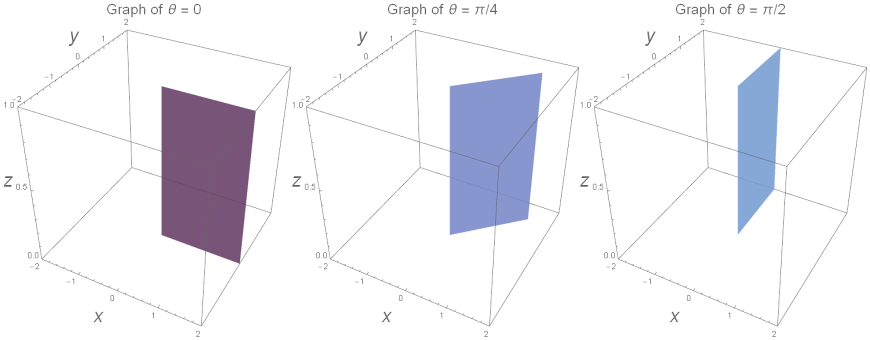
The graphs of $r = c$ are vertical cylinders $x^2 + y^2 = c^2$:





Cylindrical Coordinates, III

The graphs of $\theta = c$ are vertical half-planes:



Cylindrical Coordinates, IV

Example: Perform the following coordinate conversions:

1. Find cylindrical coordinates for $(x, y, z) = (1, 1, 3)$.
2. Find rectangular coordinates for $(r, \theta, z) = (4, \pi/6, 0)$.
3. Find cylindrical coordinates for $(x, y, z) = (-\sqrt{3}, 1, -2)$.

Cylindrical Coordinates, IV

Example: Perform the following coordinate conversions:

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2. Find rectangular coordinates for $(r, \theta, z) = (4, \pi/6, 0)$.
3. Find cylindrical coordinates for $(x, y, z) = (-\sqrt{3}, 1, -2)$.

- For $(x, y, z) = (1, 1, 3)$ we have $(r, \theta, z) = (\sqrt{2}, \pi/4, 3)$.
- For $(r, \theta, z) = (4, \pi/6, 0)$ we have $(x, y, z) = (2\sqrt{3}, 2, 0)$.
- For $(x, y, z) = (-\sqrt{3}, 1, -2)$ we have $(r, \theta, z) = (2, 5\pi/6, -2)$.

Cylindrical Coordinates, V

The parameters r and θ are essentially the same as in polar.

- Explicitly, r measures the distance of a point to the z-axis.
- Also, θ measures the angle (in a horizontal plane) from the positive x-direction.

Cylindrical coordinates are useful in simplifying regions that have a circular symmetry.

- In particular, the cylinder $x^2 + y^2 = a^2$ in 3-dimensional rectangular coordinates has the much simpler equation $r = a$ in cylindrical.
- Likewise, the cone $z = a\sqrt{x^2 + y^2}$ has the much simpler equation $z = ar$.
- More generally, $z = f(r)$ is the surface of revolution obtained by revolving the graph of $z = f(x)$ around the z-axis.

Cylindrical Coordinates, VI

The last task for cylindrical coordinates is to compute the volume differential dV.

- For $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the Jacobian is
$$J = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$
- Thus the differential in cylindrical coordinates is
$$\boxed{dV = r\,dz\,dr\,d\theta}.$$
- Notice that this is just the polar differential $dA = r\,dr\,d\theta$ with a dz in front of it.
- We typically set up cylindrical integrals with the integration order $dz\,dr\,d\theta$, since typically the z-bounds are the most complicated. But other orders are, of course, possible!

Cylindrical Coordinates, VII

Like most of our other triple integrals, the most difficult part is setting up the integral.

- When we want to set up a triple integral in cylindrical coordinates with integration order $dz\,dr\,d\theta$, we can project the solid into the xy-plane (equivalently, the $r\theta$-plane) and then set up the r and θ limits just as in polar coordinates.
- We can then find the z limits just as with triple integrals in rectangular coordinates: the lower z limit is the equation of the lower bounding surface, while the upper z limit is the equation of the upper bounding surface.

Cylindrical Coordinates, VII

<u>Example</u>: Set up and evaluate $\iiint_D \sqrt{x^2 + y^2}\, dV$ where D is the region with $0 \le z \le 3$ inside the cylinder $x^2 + y^2 = 4$.

Cylindrical Coordinates, VII

Example: Set up and evaluate $\iiint_D \sqrt{x^2+y^2}\,dV$ where D is the region with $0 \le z \le 3$ inside the cylinder $x^2+y^2=4$.

- We use cylindrical coordinates, since the region is a cylinder.
- In cylindrical coordinates, the cylinder has equation $r = 2$.
- There are no restrictions on θ, so we have $0 \le \theta \le 2\pi$, and we were given $0 \le z \le 3$.
- Since $\sqrt{x^2+y^2} = r$, the function is simply $f(r,\theta,z) = r$, and the cylindrical differential is $r\,dz\,dr\,d\theta$.
- The integral is therefore

$$\int_0^{2\pi}\int_0^2\int_0^3 r \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 3r^2 dr\,d\theta = \int_0^{2\pi} 8\,d\theta = 16\pi.$$

Summary

We discussed general changes of coordinates.

We introduced cylindrical coordinates and how to set up triple integrals in cylindrical coordinates.

We introduced spherical coordinates.

Next lecture: Cylindrical and spherical coordinates.