Math 2321 (Multivariable Calculus)

Lecture #18 of 38 $\sim$ March 3, 2021

Triple Integrals

- Triple Integrals in Rectangular Coordinates
- Setting Up Iterated Triple Integrals

This material represents §3.2 from the course notes.

Triple Integrals, I

Last time, we briefly mentioned triple integrals. Today we will discuss how to set up and evaluate triple integrals in rectangular coordinates.

- One way to interpret what a triple integral represents is to think of a function $f(x, y, z)$ as being the density of a solid object D at a given point (x, y, z).
- Then the triple integral of $f(x, y, z)$ on the region D represents the total mass of the solid.
- We will give some other uses and interpretations of triple integrals later. (Many of the applications are motivated by physics / related areas, such as computing electrical or magnetic flux.)

Iterated Integrals, II

Like with double integrals, we will write all of our triple integrals as iterated integrals.

- Computing a triple integral, once we have written it down, is usually straightforward, much like with a double integral.
- Generally, the more difficult part of the problems is setting up the integral, which requires us to sketch the region and figure out the proper bounds of integration.
- To be fair, actually computing a triple integral can involve a lot of algebra and it may take a while to do all the calculations, but there is nothing conceptually harder than what we were doing with iterated double integrals.
- Once we have the iterated integral set up, however, it's just calculation.

Iterated Integrals, III

Example: Evaluate the integral $\int_0^1 \int_0^y \int_x^y xyz \, dz \, dx \, dy$.

Iterated Integrals, III

Example: Evaluate the integral $\int_0^1 \int_0^y \int_x^y xyz\,dz\,dx\,dy$.

- We just work one step at a time, starting from the inside:

$$\begin{aligned}
\int_0^1 \int_0^y \int_x^y xyz\,dz\,dx\,dy &= \int_0^1 \int_0^y \left[\frac{1}{2}xyz^2\right]\Big|_{z=x}^{y} dx\,dy \\
&= \int_0^1 \int_0^y \left(\frac{1}{2}xy^3 - \frac{1}{2}x^3 y\right) dx\,dy \\
&= \int_0^1 \left(\frac{1}{4}x^2y^3 - \frac{1}{8}x^4 y\right)\Big|_{x=0}^{y} dy \\
&= \int_0^1 \frac{1}{8}y^5\,dy \\
&= \frac{1}{48}y^6\Big|_{y=0}^{1} = \frac{1}{48}.
\end{aligned}$$

Iterated Integrals, IV

Now we focus on how to set up triple integrals as iterated integrals.

- To do this, we choose an order of integration, and then slice up the region of integration accordingly to identify the integration bounds for each variable.
- However, since we have 3 variables instead of 2, we now have $3! = 6$ possible integration orders.
- We most commonly use the order $dz\,dy\,dx$, but depending on the problem, other integration orders may be preferable.

Iterated Integrals, V

We might also worry that the value of a triple integral might depend on the order of integration, but conveniently, we have a version of Fubini's theorem here that guarantees the value is independent of the order as long as f is continuous:

Theorem (Fubini's Theorem)

If $f(x, y, z)$ is continuous on $D = \{(x, y, z) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x),\ h_1(x, y) \leq z \leq h_2(x, y)\}$, then

$$\iiint_D f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dy\, dx,$$

and all other orders of integration will also yield the same value.

Iterated Integrals, VI

Here is the procedure for setting up triple integrals:

1. Determine the region of integration, and sketch it.
2. Decide on an order of integration and slice up the region according to the chosen order.
3. Determine the limits of integration one at a time, starting with the outer variable. The region may need to be split into several pieces, if the boundary surfaces change definition in the middle of the region.
4. Evaluate the integral.

Iterated Integrals, VII

The difficult part is identifying the limits of integration.

- The simplest method is to project the solid region into the plane spanned by the outer and middle variables, obtaining a region in that plane: then set up the outer and middle limits in the same way as for a double integral on that planar region.
- With the integration orders $dz\,dy\,dx$ or $dz\,dx\,dy$ we project into the xy-plane, with the orders $dy\,dz\,dx$ or $dy\,dx\,dz$ we project into the xz-plane, and with $dx\,dz\,dy$ or $dx\,dy\,dz$ we project into the yz-plane.
- Then, to find the bounds on the inner limit, we imagine moving parallel to the direction of the inner variable until we enter the region, and continuing until we leave the region. The "entry" surface is the lower limit, while the "exit" surface is the upper limit.

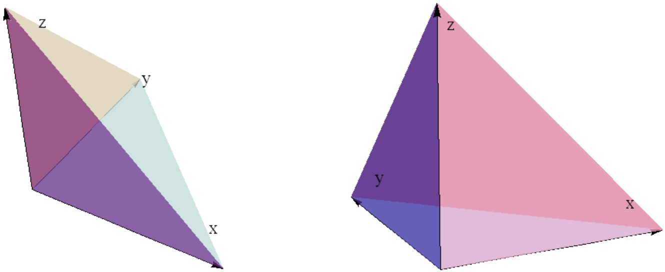
Iterated Integrals, VIII

<u>Example</u>: Find $\iiint_D x\,dV$ where D is the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

Iterated Integrals, VIII

<u>Example</u>: Find $\iiint_D x\,dV$ where D is the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

- First, we sketch the region. It is a triangular pyramid (or, if you want to be fancy, a tetrahedron) whose vertices are $(0, 0, 0)$, $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$:



Iterated Integrals, IX

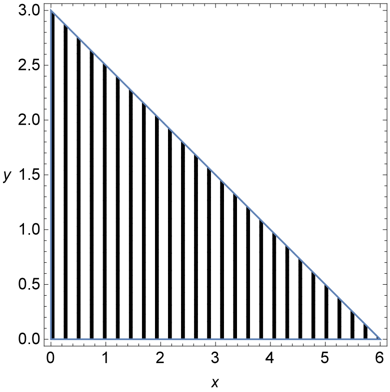
<u>Example</u>: Find $\iiint_D x \, dV$ where D is the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

- There are six possible orders of integration, and we could use any of the six.
- We will set up the integral in the order $dz\, dy\, dx$, which requires us to project this solid into the xy-plane.
- If we view the region from the top down, we can see that this projection will be a triangle.

Iterated Integrals, X

<u>Example</u>: Find $\iiint_D x\, dV$ where D is the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

Here is the projection into the xy-plane, cut into vertical slices:

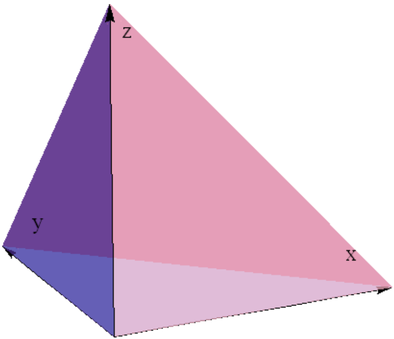


- Since we are in the plane $z = 0$, the diagonal line has equation $x + 2y = 6$.
- The slices start at $x = 0$ and end at $x = 6$.
- The bottom curve of each slice is $y = 0$ while the top curve is $y = (6 - x)/2$.
- These give us the x and y limits of integration.

Iterated Integrals, XI

Example: Find $\iiint_D x\,dV$ where D is the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

For the z-limits, we need the 3-dimensional picture of the solid.



- For fixed x and y, as we move in the direction of increasing z, we enter the solid through the xy-plane $z = 0$ and exit the solid through the tilted plane $z = (6 - x - 2y)/3$.
- Thus, the bounds on z are $0 \le z \le (6 - x - 2y)/3$.

Iterated Integrals, XII

Example: Find $\iiint_D x\,dV$ where D is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

- Putting all this together shows that the integral is
$$\int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} x\,dz\,dy\,dx$$

Iterated Integrals, XII

<u>Example</u>: Find $\iiint_D x\,dV$ where D is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + 3z = 6$.

- Putting all this together shows that the integral is

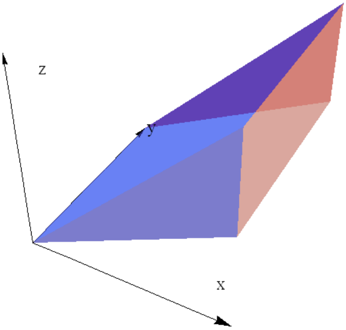
$$\int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} x\,dz\,dy\,dx$$
$$= \int_0^6 \int_0^{(6-x)/2} [xz]\,\Big|_{z=0}^{(6-x-2y)/3} dy\,dx$$
$$= \int_0^6 \int_0^{(6-x)/2} \left[2x - \tfrac{1}{3}x^2 - \tfrac{2}{3}xy\right] dy\,dx$$
$$= \int_0^6 \left[2xy - \tfrac{1}{3}x^2y - \tfrac{1}{3}xy^2\right]\Big|_{y=0}^{(6-x)/2} dx$$
$$= \int_0^6 \left[x(6-x) - \tfrac{1}{6}x^2(6-x) - \tfrac{1}{12}x(6-x)^2\right] dx$$
$$= \int_0^6 \left[3x - x^2 + \tfrac{1}{12}x^3\right] dx = \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{48}x^4\right]\Big|_{x=0}^{6} = 9.$$

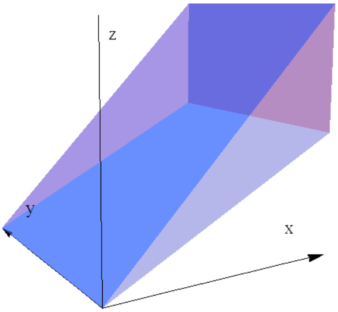
Iterated Integrals, XIII

Example: Set up an iterated integral for $\iiint_D x^2 \, dV$, where D is the "triangular wedge" bounded by the planes $x = 0$, $x = 1$, $z = x$, $z = 2x$, $y = 0$, and $y = 1$.

Iterated Integrals, XIII

Example: Set up an iterated integral for $\iiint_D x^2\, dV$, where D is the "triangular wedge" bounded by the planes $x = 0$, $x = 1$, $z = x$, $z = 2x$, $y = 0$, and $y = 1$.





Iterated Integrals, XIV

Example: Set up an iterated integral for $\iiint_D x^2 \, dV$, where D is the "triangular wedge" bounded by the planes $x = 0$, $x = 1$, $z = x$, $z = 2x$, $y = 0$, and $y = 1$.

Iterated Integrals, XIV

<u>Example</u>: Set up an iterated integral for $\iiint_D x^2 \, dV$, where D is the "triangular wedge" bounded by the planes $x = 0$, $x = 1$, $z = x$, $z = 2x$, $y = 0$, and $y = 1$.

- Note here that both x and y are bounded by constants.
- So if we project into the xy-plane, we can describe the solid very easily, since the plane region is a rectangle.
- We have $0 \leq x \leq 1$ and $0 \leq y \leq 1$, so those are our outer limits of integration.
- Looking at the solid, we can see that for specific x and y, the lower limit for z is $z = x$ and the upper limit is $z = 2x$.

Iterated Integrals, XIV

Example: Set up an iterated integral for $\iiint_D x^2 \, dV$, where D is the "triangular wedge" bounded by the planes $x = 0$, $x = 1$, $z = x$, $z = 2x$, $y = 0$, and $y = 1$.

- Note here that both x and y are bounded by constants.
- So if we project into the xy-plane, we can describe the solid very easily, since the plane region is a rectangle.
- We have $0 \le x \le 1$ and $0 \le y \le 1$, so those are our outer limits of integration.
- Looking at the solid, we can see that for specific x and y, the lower limit for z is $z = x$ and the upper limit is $z = 2x$.
- So we get the integral $\displaystyle\int_0^1 \int_0^1 \int_x^{2x} x^2 \, dz \, dy \, dx$.

Iterated Integrals, XV

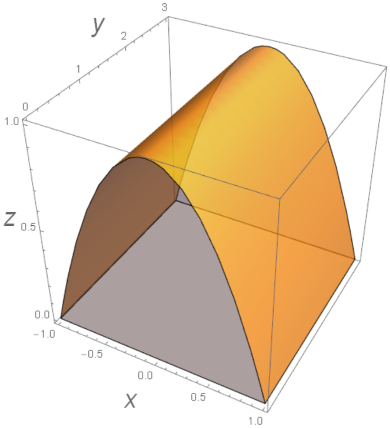
Example: Set up an iterated integral for each of the following:

1. $\iiint_D z\, dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.

Iterated Integrals, XV

<u>Example</u>: Set up an iterated integral for each of the following:

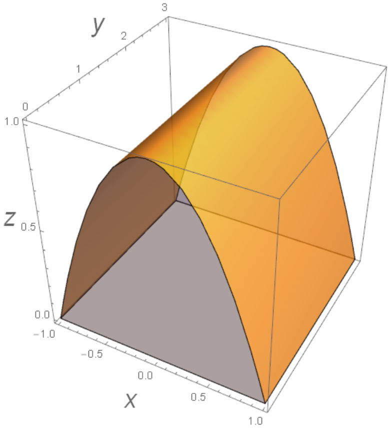
1. $\iiint_D z\,dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.



Iterated Integrals, XV

Example: Set up an iterated integral for each of the following:

1. $\iiint_D z\,dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.

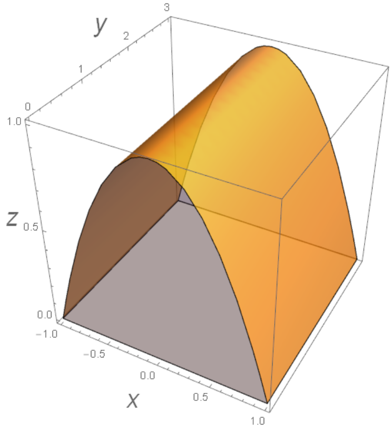


- For $dz\,dy\,dx$, we project into the xy-plane. This gives the rectangle $-1 \le x \le 1$, $0 \le y \le 3$.
- Then the lower z-limit is $z = 0$ and the upper z-limit is $z = 1 - x^2$.
- Thus, our triple integral is $\displaystyle\int_{-1}^{1}\int_{0}^{3}\int_{0}^{1-x^2} z\,dz\,dy\,dx$.

Iterated Integrals, XVI

Example: Set up an iterated integral for each of the following:

1. $\iiint_D z\, dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.

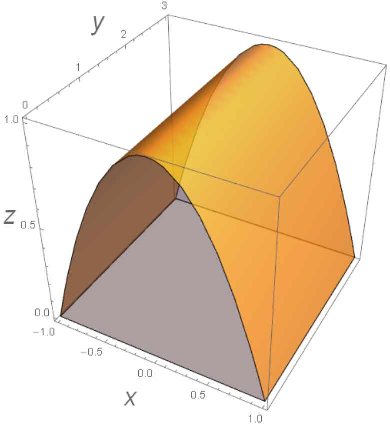


- We could also use other orders. For $dy\, dz\, dx$ we project into the xz-plane.

Iterated Integrals, XVI

Example: Set up an iterated integral for each of the following:

1. $\iiint_D z\,dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.

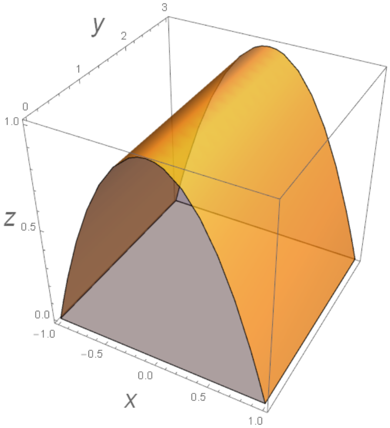


- We could also use other orders. For $dy\,dz\,dx$ we project into the xz-plane.
- The limits are $-1 \le x \le 1$, $0 \le z \le 1 - x^2$, and $0 \le y \le 3$.
- Our triple integral is $$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{3} z\,dy\,dz\,dx.$$

Iterated Integrals, XVII

Example: Set up an iterated integral for each of the following:

1. $\iiint_D z\,dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.

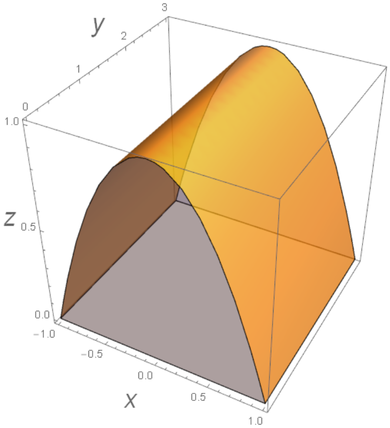


- For $dx\,dy\,dz$ we project into the yz-plane.

Iterated Integrals, XVII

Example: Set up an iterated integral for each of the following:

1. $\iiint_D z\,dV$ on the region D below the graph of $z = 1 - x^2$, above the xy-plane, with $0 \le y \le 3$.



- For $dx\,dy\,dz$ we project into the yz-plane.
- The limits are $-1 \le z \le 1$, $-1 \le y \le 1$.
- For x, we enter the solid through $x = -\sqrt{1-z}$ and leave through $x = \sqrt{1-z}$.
- Our triple integral is
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}} z\,dx\,dy\,dz.$$

Iterated Integrals, XVIII

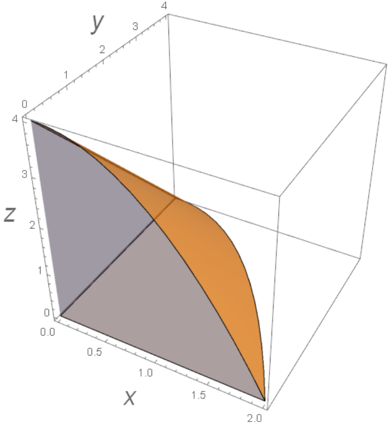
Example: Set up an iterated integral for each of the following:

2. The integral of y^3 on the region in the first octant bounded by the coordinate planes and the surface $x^2 + y + z = 4$.

Iterated Integrals, XVIII

Example: Set up an iterated integral for each of the following:

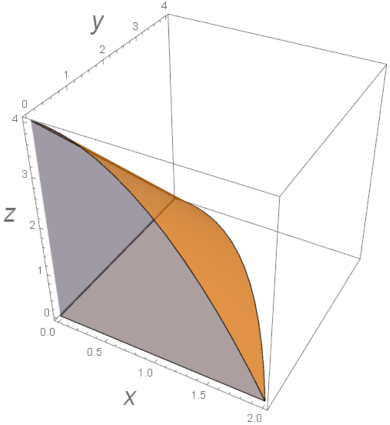
2. The integral of y^3 on the region in the first octant bounded by the coordinate planes and the surface $x^2 + y + z = 4$.



Iterated Integrals, XVIII

Example: Set up an iterated integral for each of the following:

2. The integral of y^3 on the region in the first octant bounded by the coordinate planes and the surface $x^2 + y + z = 4$.



- For $dz\,dy\,dx$, we project into the xy-plane.
- The region lies below the curve $x^2 + y = 4$. This gives limits $0 \le x \le 2$ and $0 \le y \le 4 - x^2$.
- Then the z-limits are $z = 0$ and $z = 4 - x^2 - y$.
- Thus, our triple integral is $\int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} y^3\,dz\,dy\,dx$.

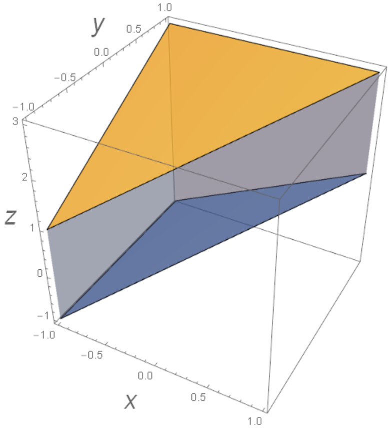
Iterated Integrals, XIX

Example: Set up an iterated integral for each of the following:

3. $\iiint_D xyz\,dV$ where D is the region with $-1 \le x \le 1$, $x \le y \le 1$, above $z = x$, and below $z = y + 2$.

Example: Set up an iterated integral for each of the following:

3. $\iiint_D xyz\, dV$ where D is the region with $-1 \le x \le 1$, $x \le y \le 1$, above $z = x$, and below $z = y + 2$.

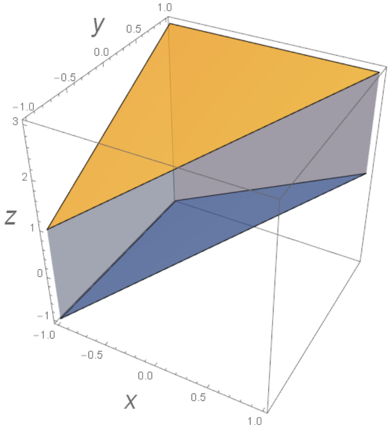


- For $dz\, dy\, dx$, we project into the xy-plane.

Iterated Integrals, XIX

Example: Set up an iterated integral for each of the following:

3. $\iiint_D xyz\, dV$ where D is the region with $-1 \le x \le 1$, $x \le y \le 1$, above $z = x$, and below $z = y + 2$.



- For $dz\, dy\, dx$, we project into the xy-plane.
- The region has $-1 \le x \le 1$ and $x \le y \le 1$, so these are our x and y limits.
- The bottom surface is $z = x$ and the top surface is $z = y + 2$.
- Thus, our triple integral is $$\int_{-1}^{1} \int_{x}^{1} \int_{x}^{y+2} xyz\, dz\, dy\, dx.$$

Miscellany, I

These problems involving setting up triple integrals can often be very tricky, because we have to have a good idea of how the surfaces intersect with each other.

- It is very easy, if we don't draw an accurate picture, to get some of the integration limits wrong. (That's why having an accurate computer-plotted graph is so important.)

We can also change the order in a triple integral.

- This is essentially the same procedure as with double integrals, except we have to draw the region in 3 dimensions rather than 2 dimensions.
- We've done enough triple integral setups today that I won't do any of these. But they are very much like the procedures we were previously using.

In fact, there exist computational algorithms that can do both of these things for us!

Miscellany, II

The procedure is known as cylindrical algebraic decomposition.

- A cylindrical algebraic decompostion converts a description of a region in space bounded by polynomial inequalities (such as $x^2 \leq y + z$ or $x^2 + y^2 + z^2 < 4$) to a union of regions $a \leq x \leq b$, $c(x) \leq y \leq d(x)$, $e(x, y) \leq z \leq f(x, y)$.
- These are precisely the types of regions we need to use in order to set up an iterated triple integral.
- For example, applying this algorithm to the region defined by the inequalities $x^2 + y^2 < 1$, $x^2 + z^2 < 1$, $y^2 + z^2 < 1$, and $0 < x < y < z$ shows that it consists of a single piece defined by $0 < x < \frac{1}{\sqrt{2}}$, $x < y < \frac{1}{\sqrt{2}}$, and $y < z < \sqrt{1 - y^2}$.

As a final remark, we will note that a cylindrical decomposition can be computed with any variable order, so we could even use it to change the order of integration.

Summary

We discussed triple integrals in rectangular coordinates.

Next lecture: More triple integrals, changes of coordinates in double and triple integrals.