Math 2321 (Multivariable Calculus)

Lecture #17 of 38 $\sim$ March 1, 2021

Double Integrals in Polar Coordinates

- Double Integrals in Polar Coordinates
- Triple Integrals

This material represents §3.3.2 + §3.2.1 from the course notes.

Polar Coordinates

Last time we briefly reviewed polar coordinates:

Definition

The *polar coordinates* (r, θ) *of a point* (x, y) *satisfy* $x = r\cos(\theta)$, $y = r\sin(\theta)$, *for* $r \geq 0$ *and* $0 \leq \theta \leq 2\pi$.

- The parameter r gives the radial distance from the origin (or "pole"), while θ measures the angle with respect to the positive x-axis.
- Some conventions allow for negative values of r. We will insist that $r \geq 0$ in our setup.
- Since sine and cosine are periodic, we implicitly identify angles θ that differ by an integral multiple of 2π radians.
- We have $r = \sqrt{x^2 + y^2}$, $\theta = \begin{cases} \tan^{-1}(y/x) \text{ for } x > 0 \\ \tan^{-1}(y/x) + \pi \text{ for } x < 0 \end{cases}$.

Integration in Polar Coordinates, I

The primary reason to use polar coordinates is that they will simplify integrals over regions that are portions of circles, because circles have simple descriptions in polar coordinates.

- Specifically, the circle $x^2 + y^2 = a^2$ in rectangular coordinates (over which it is cumbersome to set up double integrals) becomes the much simpler equation $r = a$ in polar coordinates.
- Polar coordinates are also useful in simplifying functions which involve $x^2 + y^2$ or (especially) $\sqrt{x^2 + y^2}$.
- Lines through the origin also have reasonably simple descriptions in polar: the line $y = mx$ becomes the pair of rays $\theta = \tan^{-1}(m)$ and $\theta = \tan^{-1}(m) + \pi$ when written in polar coordinates. (The two rays point in opposite directions.)

Integration in Polar Coordinates, II

Now we can describe how to set up iterated integrals in polar coordinates.

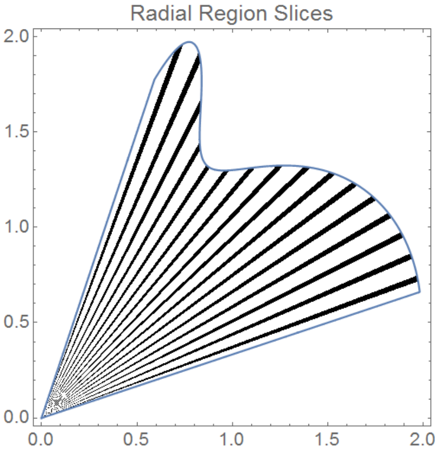
- Suppose we want to integrate the function $f(x, y)$ on the region R: this is the double integral $\iint_R f(x, y) dA$.
- To set this up as an iterated integral in polar coordinates, we typically use the integration order $dr\, d\theta$, since most of the polar curves we will work with have the form $r = f(\theta)$ or $\theta =$ constant.

There are three things we must do:

1. Convert the limits of integration to polar coordinates.
2. Convert the function to polar coordinates.
3. Convert the area differential dA to polar coordinates.

Integration in Polar Coordinates, III

Now imagine "slicing" our region radially:



Integration in Polar Coordinates, IV

Using the radial slices, we can identify the limits of integration in polar coordinates:

- The limits for θ will be the minimum and maximum values of θ: the range of values of θ where we have slices.
- The limits for r will be the minimum and maximum value of r on any given slice, in terms of θ.
- For regions that lie between two curves $r = f_{\text{inner}}(\theta)$ and $r = f_{\text{outer}}(\theta)$, the inner curve is the lower limit and the outer curve is the upper limit.

To convert the function $f(x, y)$ to polar coordinates, we simply plug in $x = r\cos\theta$ and $y = r\sin\theta$.

Integration in Polar Coordinates, V

The last task is to convert the area differential dA into polar coordinates.

- Your first guess is probably that $dA = dr\, d\theta$, in parallel to the rectangular area differential $dA = dy\, dx = dx\, dy$.
- However, this is not correct!
- To explain why, consider where the area differential comes from: it is the area of the region formed by changing the parameters x and y by small amounts δx and δy.
- The resulting shape is simply a rectangle with side lengths Δx and Δy, so we get $\Delta A = \Delta x\, \Delta y$.
- As Δx and Δy become small, the limit is $dA = dx\, dy$.

Integration in Polar Coordinates, VI

Now consider what happens if we have a radial region with radius r, and we change r by Δr and θ by $\Delta\theta$:

- The resulting shape is a radial annulus with inner radius r, outer radius $r + \Delta r$, and angle $\Delta\theta$.
- The area is

$$\begin{aligned}\Delta A = \frac{1}{2}(\Delta\theta)[(r+\Delta r)^2 - r^2] &= \frac{1}{2}\Delta\theta[2r\Delta r + (\Delta r)^2] \\ &= r\Delta r\Delta\theta + \tfrac{1}{2}(\Delta r)^2\Delta\theta.\end{aligned}$$

- As Δr and $\Delta\theta$ become small, the second term drops away, and we obtain $dA = r\,dr\,d\theta$. Note the factor of r in front!

Integration in Polar Coordinates, VII

So, putting all of this together, to set up an iterated integral $\iint_R f(x,y)\,dA$ in polar coordinates, we do the following:

1. Draw the region R. Slice it radially and use the slices to identify the polar limits of integration.
2. Convert the function $f(x,y)$ to polar coordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$.
3. Write down the polar area differential $dA = r\,dr\,d\theta$.
4. Evaluate the resulting integral.

Integration in Polar Coordinates, VIII

Example: Integrate the function $f(x, y) = x^2 + y^2$ on the region R given by the interior of the unit circle $x^2 + y^2 \leq 1$.

Integration in Polar Coordinates, VIII

Example: Integrate the function $f(x, y) = x^2 + y^2$ on the region R given by the interior of the unit circle $x^2 + y^2 \leq 1$.

- The region is defined by $r \leq 1$. Since we have no restrictions on θ, we want $0 \leq \theta \leq 2\pi$.
- Since r is always nonnegative, our limits for r are $0 \leq r \leq 1$.
- The function is $f(r\cos\theta, r\sin\theta) = \cos^2\theta + \sin^2\theta = 1$.
- The differential is $dA = r\,dr\,d\theta$.

Integration in Polar Coordinates, VIII

Example: Integrate the function $f(x, y) = x^2 + y^2$ on the region R given by the interior of the unit circle $x^2 + y^2 \le 1$.

- The region is defined by $r \le 1$. Since we have no restrictions on θ, we want $0 \le \theta \le 2\pi$.
- Since r is always nonnegative, our limits for r are $0 \le r \le 1$.
- The function is $f(r\cos\theta, r\sin\theta) = \cos^2\theta + \sin^2\theta = 1$.
- The differential is $dA = r\,dr\,d\theta$.
- Then the integral is $\int_0^{2\pi}\int_0^1 (r^2) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$

$$= \int_0^{2\pi} \frac{1}{4} r^4 \Big|_{r=0}^{1} d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}.$$

Integration in Polar Coordinates, IX

Example: Integrate the function $f(x, y) = x^2 + y^2$ on the region R given by the interior of the unit circle $x^2 + y^2 \leq 1$.

Integration in Polar Coordinates, IX

Example: Integrate the function $f(x,y) = x^2 + y^2$ on the region R given by the interior of the unit circle $x^2 + y^2 \le 1$.

- We could have done this problem in rectangular coordinates. Using the integration order $dy\,dx$, here is how that goes:

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx &= \int_{-1}^{1} \left(x^2y + \tfrac{1}{3}y^3\right)\Big|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx \\
&= \int_{-1}^{1} 2\left(x^2\sqrt{1-x^2} - \tfrac{1}{3}(1-x^2)^{3/2}\right)dx \\
&= \tfrac{1}{6}\left[(x+2x^3)\sqrt{1-x^2} + 3\sin^{-1}x\right]\Big|_{x=-1}^{1} \\
&= \pi/2.
\end{aligned}$$

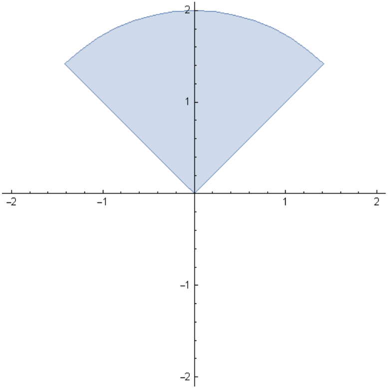
Note that there is an integration by parts and some trigonometric substitutions (omitted!) needed to get from line 2 to line 3.

Integration in Polar Coordinates, X

Example: Integrate $f(x, y) = x + 2y$ on the region R lying above the lines $y = x$ and $y = -x$ and inside the circle $x^2 + y^2 = 4$.

Integration in Polar Coordinates, X

Example: Integrate $f(x, y) = x + 2y$ on the region R lying above the lines $y = x$ and $y = -x$ and inside the circle $x^2 + y^2 = 4$. This region is a quarter-disc:



Integration in Polar Coordinates, XI

Example: Integrate $f(x, y) = x + 2y$ on the region R lying above the lines $y = x$ and $y = -x$ and inside the circle $x^2 + y^2 = 4$.

Integration in Polar Coordinates, XI

Example: Integrate $f(x, y) = x + 2y$ on the region R lying above the lines $y = x$ and $y = -x$ and inside the circle $x^2 + y^2 = 4$.

- To find the limits of integration, we convert the equations for the boundary into polar coordinates and use the picture.
- The line $y = x$ gives the right boundary $\theta = \pi/4$ and the line $y = -x$ gives the left boundary $\theta = 3\pi/4$.
- The circle $x^2 + y^2 = 4$ becomes $r = 2$.
- Thus, our limits are $\pi/4 \le \theta \le 3\pi/4$ and $0 \le r \le 2$.
- The function is $f(x, y) = x + 2y = r\cos\theta + 2r\sin\theta$, while the area differential, as always, is $dA = r\,dr\,d\theta$.
- So the integral is $\displaystyle\int_{\pi/4}^{3\pi/4}\int_0^2 (r\cos\theta + 2r\sin\theta)\cdot r\,dr\,d\theta$.

Integration in Polar Coordinates, XII

Example: Integrate $f(x, y) = x + 2y$ on the region R lying above the lines $y = x$ and $y = -x$ and inside the circle $x^2 + y^2 = 4$.

Integration in Polar Coordinates, XII

Example: Integrate $f(x, y) = x + 2y$ on the region R lying above the lines $y = x$ and $y = -x$ and inside the circle $x^2 + y^2 = 4$.

- Now we just have to evaluate the integral:

$$\begin{aligned}
\int_{\pi/4}^{3\pi/4} \int_0^2 (r\cos\theta + 2r\sin\theta) \cdot r\,dr\,d\theta &= \int_{\pi/4}^{3\pi/4} (\cos\theta + 2\sin\theta) \cdot \frac{1}{3} r^3 \Big|_{r=0}^{2} d\theta \\
&= \int_{\pi/4}^{3\pi/4} \frac{8}{3} (\cos\theta + 2\sin\theta)\,d\theta \\
&= \frac{8}{3} (-\sin\theta + 2\cos\theta) \Big|_{\theta=\pi/4}^{3\pi/4} = \frac{8\sqrt{2}}{3}.
\end{aligned}$$

Integration in Polar Coordinates, XIII

We can also convert integrals that have been set up in rectangular coordinates to polar coordinates.

- Of course, usually we only want to do this when the integral will be easier to evaluate in polar coordinates.
- An iterated integral will be easier to evaluate in polar coordinates when the region and function both have reasonably nice descriptions in polar.
- Some obvious signs suggesting polar coordinates are if the function involves $\sqrt{x^2 + y^2}$ terms, or if the region turns out to be a portion of a circle.

Integration in Polar Coordinates, XIV

Example: Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

Integration in Polar Coordinates, XIV

Example: Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

- Here, the function involves $\sqrt{x^2+y^2}$ (in fact, that *is* the function!) and the region is the interior of the circle $x^2+y^2=4$, so we will switch to polar coordinates.
- In polar coordinates, the bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le 2$, with function $f(x,y)=\sqrt{x^2+y^2}=r$ and differential $dA = r\, dr\, d\theta$.
- So, in polar, the integral is
$$\int_0^{2\pi}\int_0^2 r \cdot r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta = \int_0^{2\pi} \frac{8}{3} d\theta = \frac{16\pi}{3}.$$

Integration in Polar Coordinates, XV

Example: Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\, dy\, dx$.

Integration in Polar Coordinates, XV

Example: Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\, dy\, dx$.

- As before we have various signs (from both the function and the region) suggesting that we should try polar coordinates.
- In polar coordinates, the bounds are $0 \le \theta \le \pi/2$ and $0 \le r \le 1$, with function $f(x,y) = e^{\sqrt{x^2+y^2}} = e^r$ and differential $dA = r\, dr\, d\theta$.
- So, in polar, the integral is $\int_0^{\pi/2} \int_0^1 e^r \cdot r\, dr\, d\theta =$

$$\int_0^{\pi/2} [r\, e^r - e^r]\Big|_{r=0}^{1} d\theta = \int_0^{\pi/2} 1\, d\theta = \pi/2.$$

Integration in Polar Coordinates, XVI

As an application of integration in polar coordinates, we can evaluate the famous Gaussian integral $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

- This integral is quite difficult to compute because the function e^{-x^2} does not have an elementary antiderivative. Even using a Taylor series approach (i.e., writing e^{-x^2} as a power series in x) does not work, because the integral is improper.
- This integral is fundamental in statistics, since $p(x) = e^{-x^2}$ arises (after a change of variables) as the probability density function of the Gaussian normal distribution.
- The normal distribution describes the distributions of quantities arising as the sum of independent small variations, such as human heights, errors in measurements, exam grades, and many other physical phenomena.
- To learn more, take Math 3081 (Probability and Statistics)! (Unrelated fun fact: I'm teaching it in summer 2.)

Integration in Polar Coordinates, XVII

Here is how to compute $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$:

- First, we can also write $I = \int_{-\infty}^{\infty} e^{-y^2}\, dy$. Multiplying gives $I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}\, dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}\, dy\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$.
- Now convert to polar coordinates: the region for this last integral is the entire plane, with integration bounds $0 \le \theta \le 2\pi$ and $0 \le r < \infty$.
- The function is $e^{-(x^2+y^2)} = e^{-r^2}$, and of course $dA = r\, dr\, d\theta$.
- Thus, in polar coordinates we see $I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta$.
- We can evaluate the polar integral using a substitution $u = r^2$ to see $I^2 = \displaystyle\int_0^{2\pi} \left[\frac{1}{2}e^{-r^2}\right]\Big|_{r=0}^{\infty} d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$.
- Therefore, since $I > 0$, we deduce that $I = \sqrt{\pi}$.

Integration in Polar Coordinates, XVIII

Example: Find the volume underneath the graph of $z = 16 - x^2 - y^2$ that lies above the xy-plane.

Integration in Polar Coordinates, XVIII

Example: Find the volume underneath the graph of $z = 16 - x^2 - y^2$ that lies above the xy-plane.

- This volume is given as a double integral $\iint_R (16 - x^2 - y^2)\, dA$ where R is the region in the plane where the surface $z = 16 - x^2 - y^2$ lies above the xy-plane.
- The region R is where $16 - x^2 - y^2 \geq 0$, which is to say, where $x^2 + y^2 \leq 16$.
- Since this is the interior of a circle, this integral will be easiest to set up in polar coordinates.

Integration in Polar Coordinates, XIX

Example: Find the volume underneath the graph of $z = 16 - x^2 - y^2$ that lies above the xy-plane.

Integration in Polar Coordinates, XIX

Example: Find the volume underneath the graph of $z = 16 - x^2 - y^2$ that lies above the xy-plane.

- The region R, in polar, is $0 \le \theta \le 2\pi$ and $0 \le r \le 4$.
- The function is $f(x, y) = 16 - x^2 - y^2 = 16 - r^2$, and as always the polar area differential is $dA = r\,dr\,d\theta$.
- Thus, the volume integral is

$$\begin{aligned}
\int_0^{2\pi} \int_0^4 (16 - r^2) r\,dr\,d\theta &= \int_0^{2\pi} \int_0^4 (16r - r^3) dr\,d\theta \\
&= \int_0^{2\pi} \left(8r^2 - \tfrac{1}{4} r^4\right)\Big|_{r=0}^{4} \\
&= \int_0^{2\pi} 64 d\theta = 128\pi.
\end{aligned}$$

Integration in Polar Coordinates, XX

Example: Evaluate the double integral $\iint_R \frac{x^2}{x^2 + y^2}\, dA$ where R is the region $2 \le x^2 + y^2 \le 3$ where $x > 0$.

Integration in Polar Coordinates, XX

Example: Evaluate the double integral $\iint_R \frac{x^2}{x^2+y^2}\, dA$ where R is the region $2 \le x^2 + y^2 \le 3$ where $x > 0$.

- The region R, in polar, is the right half of the annulus between the circles $r = \sqrt{2}$ and $r = \sqrt{3}$.
- The portion with $x > 0$ corresponds to $-\pi/2 \le \theta \le \pi/2$. (Here it is convenient to use negative θ to avoid splitting the region into 2 pieces.)
- The function is $\frac{x^2}{x^2+y^2} = \frac{r^2 \cos^2\theta}{r^2} = \cos^2\theta$, and $dA = r\,dr\,d\theta$.
- We get $\int_{-\pi/2}^{\pi/2}\int_{\sqrt{2}}^{\sqrt{3}} \cos^2\theta \cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \cos^2\theta = \frac{\pi}{2}$.

Triple Integrals, I

For the last week, we have been discussing double integrals. Now we bump our discussion into 3 dimensions with triple integrals.

- Like with double integrals, we outline the fundamental definition using Riemann sums.
- Then (next time) we will discuss how to set up and evaluate triple integrals as iterated integrals.
- After that, we will explain how to do general coordinate changes, and then talk about two very useful 3-dimensional generalizations of polar coordinates: cylindrical coordinates and spherical coordinates.

Triple Integrals, II

So, now we want to integrate functions $f(x, y, z)$ over regions in 3-space instead of functions $f(x, y)$ over regions in the plane.

- For clarity, we will use D to denote solid regions in 3-space, and reserve R for regions in the plane.
- The motivating problem for integration in three variables is somewhat less clear, however.
- For single integrals we wanted to find the area under a curve $y = f(x)$, and for double integrals we wanted to find the volume under a surface $z = f(x, y)$.
- For triple integrals, it is somewhat harder to envision what happens when we move up by 1 dimension: we would then be finding "the 4-dimensional volume under a 3-dimensional hypersurface" (whatever that means!).

Triple Integrals, III

One way to interpret what a triple integral represents is to think of a function $f(x, y, z)$ as being the density of a solid object D at a given point (x, y, z).

- Then the triple integral of $f(x, y, z)$ on the region D represents the total mass of the solid.
- We will give some other uses and interpretations of triple integrals later. (Many of the applications are motivated by physics / related areas, such as computing electrical or magnetic flux.)

Riemann Sums, I

We formalize things using Riemann sums.

Definition

For a region D in 3-space, a <u>partition</u> of D into n pieces is a list of disjoint rectangular boxes inside D, where the kth rectangle contains the point (x_k, y_k, z_k), has length Δx_k, width Δy_k, height Δz_k, and volume $\Delta V_k = \Delta z_k \cdot \Delta y_k \cdot \Delta x_k$.

The <u>norm</u> of the partition P is the largest number among the dimensions of all of the boxes in P.

Then, for a continous function $f(x, y, z)$ and a partition P of the region D, we define the <u>Riemann sum</u> of $f(x, y, z)$ on D corresponding to P to be $RS_P(f) = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k$.

Riemann Sums, II

The idea now is that we can define the triple integral of $f(x, y, z)$ on D by taking an appropriate limit of Riemann sums:

Definition

For $f(x, y, z)$ a continuous function, we define the (triple) integral of f on the region R, $\iiint\limits_D f(x, y, z)\, dV$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\mathrm{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.

This is essentially the same definition that we had for double integrals. The value $\iiint_D f(x, y, z)\, dV$, roughly speaking, is the limit of the Riemann sums of f on D, as the size of the subregions in the partition becomes small.

Riemann Sums, III

For C an arbitrary constant and $f(x, y, z)$ and $g(x, y, z)$ continuous functions, the following properties hold:

1. Integral of constant: $\iiint_D C\, dV = C \cdot \text{Volume}(D)$.
2. Constant multiple of a function:
   $\iiint_D C\, f(x, y, z)\, dV = C \cdot \iiint_D f(x, y, z)\, dV$.
3. Addition of functions:
   $\iiint_D f(x, y, z)\, dV + \iiint_D g(x, y, z)\, dV =$
   $\iiint_D [f(x, y, z) + g(x, y, z)]\, dV$.
4. Subtraction of functions:
   $\iiint_D f(x, y, z)\, dV - \iiint_D g(x, y, z)\, dV =$
   $\iiint_D [f(x, y, z) - g(x, y, z)]\, dV$.
5. Nonnegativity: if $f(x, y, z) \geq 0$, then $\iiint_D f(x, y, z)\, dV \geq 0$.
6. Union: If D_1 and D_2 don't overlap and have union D,
   $\iiint_{D_1} f(x, y, z)\, dV + \iiint_{D_2} f(x, y, z)\, dV = \iiint_D f(x, y, z)\, dV$.

Iterated Triple Integrals, I

Like with double integrals, we will write all of our triple integrals as iterated integrals.

- Computing a triple integral, once we have written it down, is usually straightforward, much like with a double integral.
- Generally, the more difficult part of the problems is setting up the integral, which requires us to sketch the region and figure out the proper bounds of integration.
- To be fair, actully computing a triple integral can involve a lot of algebra and it may take a while to do all the calculations, but there is nothing conceptually harder than what we were doing with iterated double integrals.
- Once we have the iterated integral set up, however, it's just calculation.

To finish today, let's work through the evaluation of an iterated triple integral.

Iterated Triple Integrals, II

Example: Evaluate the integral $\int_0^1 \int_0^2 \int_1^3 4xz \, dz \, dy \, dx$.

Iterated Triple Integrals, II

Example: Evaluate the integral $\int_0^1 \int_0^2 \int_1^3 4xz \, dz \, dy \, dx$.

- We just work one step at a time, starting from the inside:

$$\begin{aligned}
\int_0^1 \int_0^2 \int_1^3 4xz \, dz \, dy \, dx &= \int_0^1 \int_0^2 \left(2xz^2\right)\Big|_{z=1}^{3} \, dy \, dx \\
&= \int_0^1 \int_0^2 16x \, dy \, dx \\
&= \int_0^1 \left(16xy\right)\Big|_{y=0}^{2} \, dx \\
&= \int_0^1 32x \, dx \\
&= \left(16x^2\right)\Big|_{x=0}^{1} = 16.
\end{aligned}$$

Summary

We discussed double integrals in polar coordinates.

We introduced triple integrals.

Next lecture: Iterated triple integrals.