Math 2321 (Multivariable Calculus)

Lecture #16 of 38 $\sim$ February 25, 2021

Computing Double Integrals

- Double Integrals on General Regions
- Changing Order of Integration

This material represents §3.1.2-3.1.3 from the course notes.

Recall

Last time, we discussed how to find the volume underneath $z = f(x, y)$ above a region R in the plane.

- We gave two ways of computing this volume as an iterated integral with either integration order $dy\,dx$ or $dx\,dy$.
- With order $dy\,dx$ we have $\int_a^b \int_{c(x)}^{d(x)} f(x, y)\,dy\,dx$.
- With order $dx\,dy$ we have $\int_c^d \int_{a(y)}^{b(y)} f(x, y)\,dx\,dy$.

These will always give the same value as long as f is continuous:

Theorem (Fubini's Theorem on General Regions)

If $f(x, y)$ is continuous on a region
$R = \{(x, y) :\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ *and*
$R = \{(x, y) :\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, *then*

$$\iint_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dx\,dy.$$

Computing Double Integrals, I

Here is the procedure for setting up double integrals:

1. Determine the region of integration, and sketch it.
2. Decide on an order of integration and slice up the region according to the chosen order: vertical slices correspond to $dy\,dx$ and horizontal slices correspond to $dx\,dy$.
3. Determine the limits of integration one at a time, starting with the outer variable. The region may need to be split into several pieces, if the boundary curves change definition in the middle of the region.
   - For $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, the integral is $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.
   - For $R = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, the integral is $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.
4. Evaluate the integral.

Computing Double Integrals, II

Example: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$ with

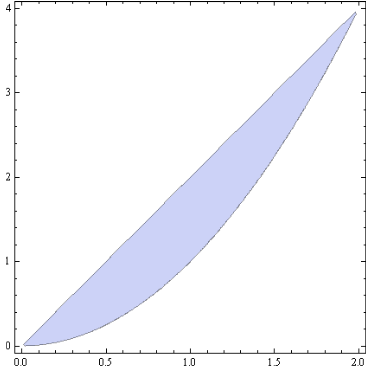
1. integration order $dy\,dx$.
2. integration order $dx\,dy$.

Computing Double Integrals, II

Example: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$ with

1. integration order $dy\,dx$.
2. integration order $dx\,dy$.

- The curves $y = 2x$ and $y = x^2$ will intersect at $(0, 0)$ and $(2, 4)$, so the region looks like this:



Computing Double Integrals, III

<u>Example</u>: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

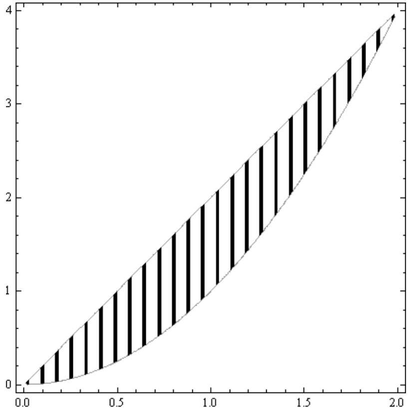
1. with integration order $dy\,dx$.

Computing Double Integrals, III

Example: Set up the integral of $f(x,y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

1. with integration order $dy\,dx$.

We slice up the region with vertical slices:



Computing Double Integrals, IV

<u>Example</u>: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

1. with integration order $dy\,dx$.

Computing Double Integrals, IV

Example: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

1. with integration order $dy\,dx$.

- The leftmost slice occurs at $x = 0$ while the rightmost slice occurs for $x = 2$.
- For any given slice, the values of y range from the lower curve $y = x^2$ to the upper curve $y = 2x$.
- So, the integral is $\int_0^2 \int_{x^2}^{2x} xy^2\,dy\,dx$.
- We compute $\int_0^2 \int_{x^2}^{2x} xy^2\,dy\,dx = \int_0^2 \left[\frac{1}{3}xy^3\right]\Big|_{y=x^2}^{2x}\,dx$ $= \int_0^2 \left[\frac{8}{3}x^4 - \frac{1}{3}x^7\right]\,dx = \left[\frac{8}{15}x^5 - \frac{1}{24}x^8\right]\Big|_{x=0}^{2} = \frac{8 \cdot 2^5}{15} - \frac{2^8}{24} = \dfrac{32}{5}$.

Computing Double Integrals, V

Example: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

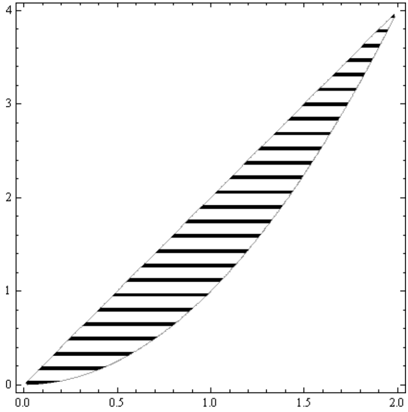
2. with integration order $dx\,dy$.

Computing Double Integrals, V

Example: Set up the integral of $f(x,y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

2. with integration order $dx\,dy$.

We slice up the region with horizontal slices:



Computing Double Integrals, VI

Example: Set up the integral of $f(x, y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

2. with integration order $dx\,dy$.

Computing Double Integrals, VI

Example: Set up the integral of $f(x,y) = xy^2$ over the finite region R between the curves $y = 2x$ and $y = x^2$

2. with integration order $dx\,dy$.

- The bottom slice occurs at $y = 0$ while the top slice occurs for $y = 4$.
- For any given slice, the values of x range from the left curve $y = 2x$ to the right curve $y = x^2$.
- Thus, the integral is $\int_0^4 \int_{y/2}^{\sqrt{y}} xy^2\,dx\,dy$.
- We compute $\int_0^4 \int_{y/2}^{\sqrt{y}} xy^2\,dx\,dy = \int_0^4 \left[\frac{1}{2}x^2y^2\right]\Big|_{x=y/2}^{\sqrt{y}}\,dy = \int_0^4 \left[\frac{1}{2}y^3 - \frac{1}{8}y^4\right]\,dy = \left[\frac{1}{8}y^4 - \frac{1}{40}y^5\right]\Big|_{y=0}^{4} = \frac{4^4}{8} - \frac{4^5}{40} = \frac{32}{5}$.

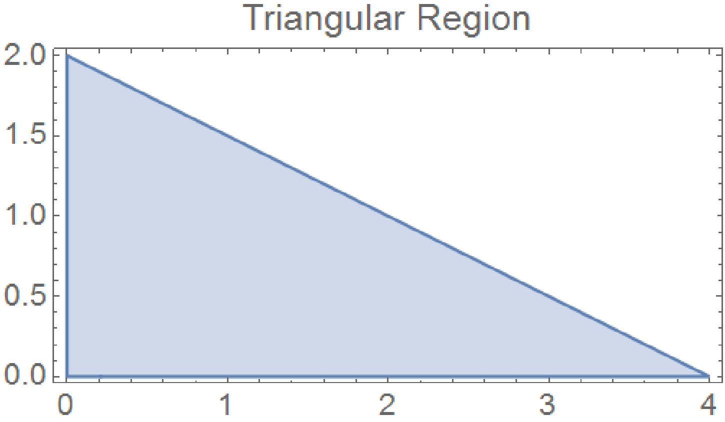
Computing Double Integrals, VII

<u>Example</u>: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

Computing Double Integrals, VII

<u>Example</u>: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

- First, we sketch the region, and then we slice it up:



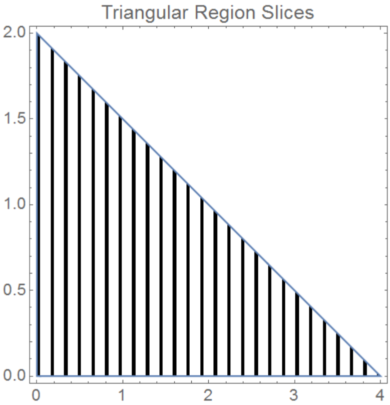
Computing Double Integrals, VIII

Example: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

Computing Double Integrals, VIII

Example: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

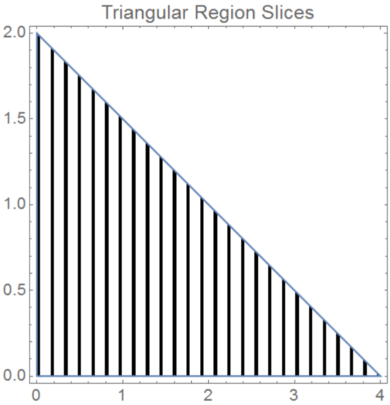
- First, vertical slices (order $dy\,dx$):



Computing Double Integrals, VIII

Example: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

- First, vertical slices (order $dy\,dx$):



- The leftmost slice occurs at $x = 0$ and the rightmost slice occurs at $x = 4$.
- For each slice, the bottom is $y = 0$ and the top is the line $x + 2y = 4$, which (since we need y as a function of x) is $y = (4 - x)/2$.
- Therefore, the double integral is $\displaystyle\int_0^4 \int_0^{(4-x)/2} xy\,dy\,dx$.

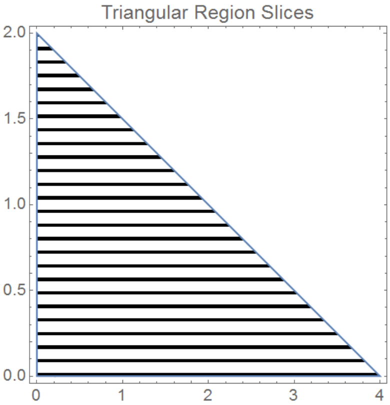
Computing Double Integrals, IX

Example: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

Computing Double Integrals, IX

Example: Set up the integral of $f(x, y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

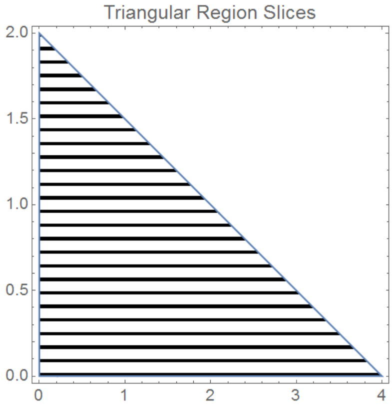
- Second, horizontal slices (order $dx\, dy$):



Computing Double Integrals, IX

Example: Set up the integral of $f(x,y) = xy$ on the region with $x, y \geq 0$ and $x + 2y \leq 4$ with both possible integration orders.

- Second, horizontal slices (order $dx\,dy$):



- The bottom slice occurs at $y = 0$ and the top slice occurs at $y = 2$.
- For each slice, the left curve is $x = 0$ and the right curve is the line $x + 2y = 1$, which (since we need x as a function of y) is $x = 1 - 2y$.
- Therefore, the double integral is $\displaystyle\int_0^2 \int_0^{4-2y} xy\,dx\,dy$.

Computing Double Integrals, X

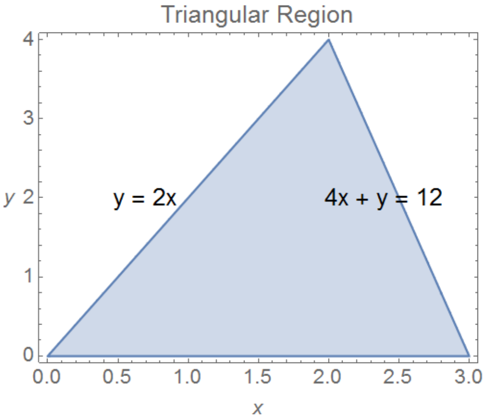
If the equation for one of the boundary curves changes in the middle of the region, we must split the region into pieces, corresponding to the different functions that make up the boundary curve.

Integrals on General Regions, XIV

Example: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0, 0)$, $(3, 0)$, and $(2, 4)$.

Integrals on General Regions, XIV

Example: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.



The equations of the two lines are $y = 2x$ and $4x + y = 12$.

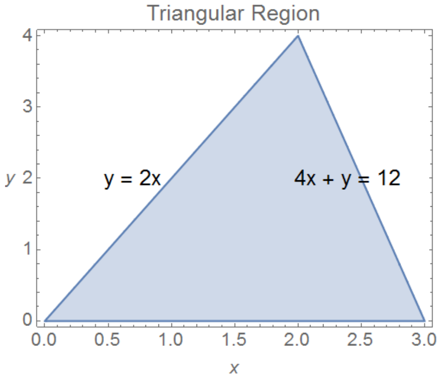
Computing Double Integrals, XI

<u>Example</u>: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

Computing Double Integrals, XI

Example: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

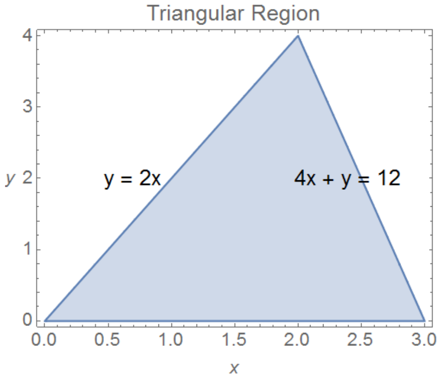
- First, vertical slices (order $dy\,dx$):



Computing Double Integrals, XI

Example: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

- First, vertical slices (order $dy\,dx$):

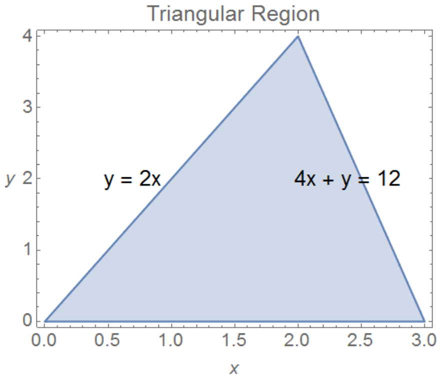


- Although the vertical slices go from $x = 0$ to $x = 3$, the identity of the top curve changes from $y = 2x$ to $4x + y = 12$ when $x = 2$.
- So, we need to split the region into two pieces, one from $0 \le x \le 2$ and the other from $2 \le x \le 3$.

Computing Double Integrals, XII

Example: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

- First, vertical slices (order $dy\,dx$):



- For $0 \le x \le 2$ the bottom curve is $y = 0$ and the top curve is $y = 2x$. This gives $\int_0^2 \int_0^{2x} x^2 y \, dy \, dx$.
- For $2 \le x \le 3$ the bottom curve is $y = 0$ and the top curve is $y = 12 - 4x$. This gives $\int_2^3 \int_0^{12-4x} x^2 y \, dy \, dx$.

The integral we want is then the sum of these two integrals.

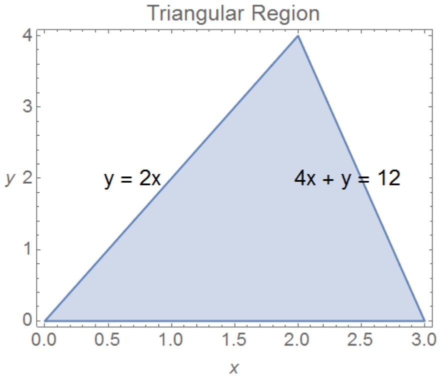
Computing Double Integrals, XIII

<u>Example</u>: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

Computing Double Integrals, XIII

Example: Set up an iterated integral for $\iint_R x^2 y \, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

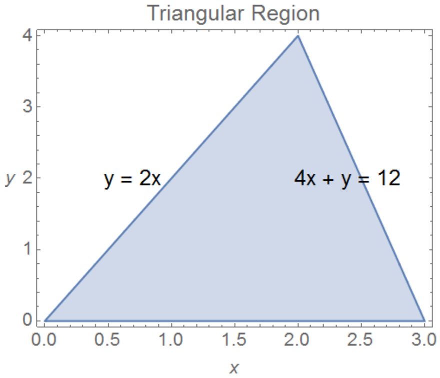
- Now we do horizontal slices (order $dx\,dy$):



Computing Double Integrals, XIII

Example: Set up an iterated integral for $\iint_R x^2 y\, dA$ on the triangular region R with vertices $(0,0)$, $(3,0)$, and $(2,4)$.

- Now we do horizontal slices (order $dx\, dy$):



- The slices go from $y = 0$ to $y = 4$.
- The left curve is $x = y/2$ and the right curve is $x = (12 - y)/4$ throughout the region.
- Thus, the desired integral is $\displaystyle\int_0^4 \int_{y/2}^{(12-y)/4} x^2 y\, dx\, dy$.

Changing Order of Integration, I

Although Fubini's Theorem guarantees that integrating in either order will yield the same result, it can happen that one order of integration is easier to evaluate than the other.

- In such situations, reversing the order of integration can be useful.
- To do this, we sketch the region, and then slice it up in the opposite direction.
- This may require splitting the region into multiple pieces, depending on the boundary curves.

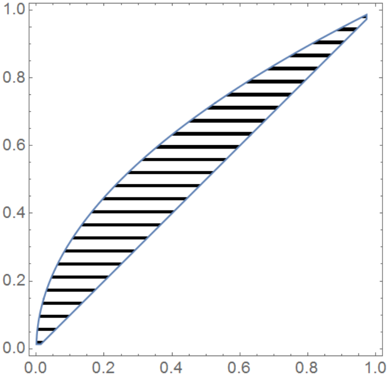
Changing Order of Integration, II

<u>Example</u>: Reverse the order of integration for $\displaystyle\int_0^1 \int_{y^2}^{y} xy\, dx\, dy$.

Changing Order of Integration, II

Example: Reverse the order of integration for $\int_0^1 \int_{y^2}^{y} xy\,dx\,dy$.

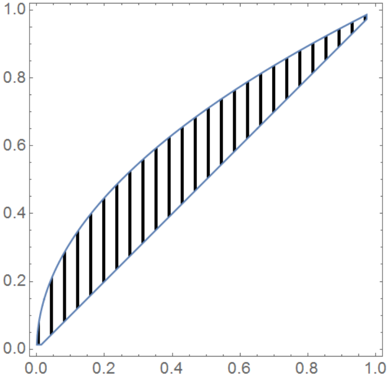
- The region is defined by $0 \le y \le 1$ and $y^2 \le x \le y$, and the current order of integration has horizontal slices:



Changing Order of Integration, III

Example: Reverse the order of integration for $\displaystyle\int_0^1 \int_{y^2}^{y} xy \, dx \, dy$.

- To reverse the order of integration, we want to slice up the region perpendicular to the x-axis so that it has vertical slices:



Changing Order of Integration, III

Example: Reverse the order of integration for $\displaystyle\int_0^1 \int_{y^2}^{y} xy\,dx\,dy$.

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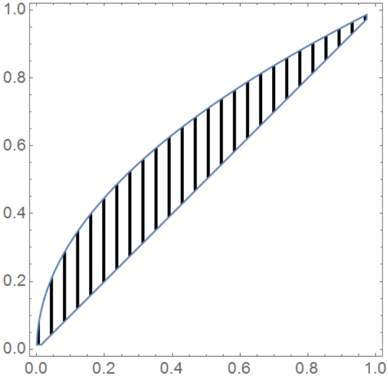


- The slices go from $x = 0$ to $x = 1$.
- The top curve is $y = \sqrt{x}$ and the bottom curve is $y = x$.
- Thus, the new integral is $\displaystyle\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$.

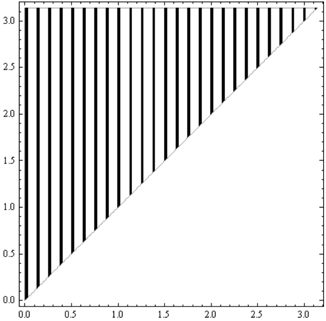
Changing Order of Integration, IV

Example: Evaluate the integral $\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y} \, dy \, dx$ by reversing the order of integration.

Changing Order of Integration, IV

Example: Evaluate the integral $\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y}\, dy\, dx$ by reversing the order of integration.

- It is not possible to evaluate the inner integral as written since $\frac{\sin(y)}{y}$ does not have an elementary antiderivative.
- The region is defined by $0 \le x \le x$ and $x \le y \le \pi$, and the current order of integration has vertical slices:



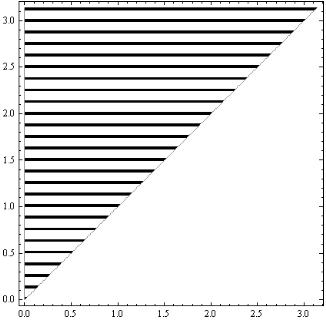
Changing Order of Integration, V

Example: Evaluate the integral $\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y}\, dy\, dx$ by reversing the order of integration.

Changing Order of Integration, V

Example: Evaluate the integral $\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y}\, dy\, dx$ by reversing the order of integration.

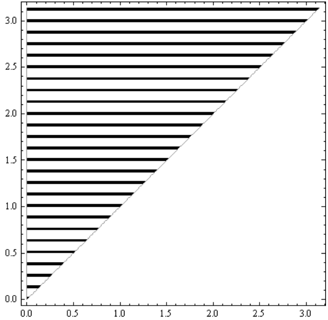
- With horizontal slices, we obtain the following:



Changing Order of Integration, V

Example: Evaluate the integral $\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y} \, dy \, dx$ by reversing the order of integration.

- With horizontal slices, we obtain the following:



- The slices go from $y = 0$ to $y = \pi$.
- The left curve is $x = 0$ and the right curve is $x = y$.
- The new integral is $\int_0^{\pi} \int_0^{y} \frac{\sin(y)}{y} \, dx \, dy$
  $= \int_0^{\pi} \sin(y) \, dy = 2.$

Polar Coordinates, I

To lead into the next lecture (double integrals in polar coordinates), we will spend the last few minutes with a quick review of polar coordinates.

- Polar coordinates are perhaps (probably? maybe?) familiar to you already, as they are often discussed in precalculus and single-variable calculus.
- Double integrals in polar coordinates are similar to double integrals in rectangular coordinates: the only difference is that we will now set up iterated integrals using the polar coordinate variables r and θ rather than the rectangular coordinates x and y.

Polar Coordinates, II

So, a brief review of polar coordinates:

Definition

The <u>polar coordinates</u> (r, θ) of a point (x, y) satisfy $x = r\cos(\theta)$, $y = r\sin(\theta)$, for $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

- The parameter r gives the radial distance from the origin (or "pole"), while θ measures the angle with respect to the positive x-axis.
- Some conventions allow for negative values of r. We will insist that $r \geq 0$ in our setup.
- Since sine and cosine are periodic, we implicitly identify angles θ that differ by an integral multiple of 2π radians.
- We have $r = \sqrt{x^2 + y^2}$, $\theta = \begin{cases} \tan^{-1}(y/x) \text{ for } x > 0 \\ \tan^{-1}(y/x) + \pi \text{ for } x < 0 \end{cases}$.

Polar Coordinates, III

Example: Perform the following coordinate conversions:

1. Find polar coordinates for $(x, y) = (1, 1)$.
2. Find rectangular coordinates for $(r, \theta) = (4, \pi/6)$.
3. Find polar coordinates for $(x, y) = (-\sqrt{3}, 1)$.
4. Find rectangular coordinates for $(r, \theta) = (2.8, 0.7)$.
5. Find polar coordinates for $(x, y) = (-6.0, -1.1)$.

Polar Coordinates, III

Example: Perform the following coordinate conversions:

1. Find polar coordinates for $(x, y) = (1, 1)$.
2. Find rectangular coordinates for $(r, \theta) = (4, \pi/6)$.
3. Find polar coordinates for $(x, y) = (-\sqrt{3}, 1)$.
4. Find rectangular coordinates for $(r, \theta) = (2.8, 0.7)$.
5. Find polar coordinates for $(x, y) = (-6.0, -1.1)$.

- For $(x, y) = (1, 1)$ we have $(r, \theta) = (\sqrt{2}, \pi/4)$.
- For $(r, \theta) = (4, \pi/6)$ we have $(x, y) = (2\sqrt{3}, 2)$.
- For $(x, y) = (-\sqrt{3}, 1)$ we have $(r, \theta) = (2, 5\pi/6)$.
- For $(r, \theta) = (2.8, 0.7)$ we have $(x, y) \approx (2.1416, 1.8038)$.
- For $(x, y) = (-6.7, -2.2)$ we have $(r, \theta) \approx (6.1, 3.4588)$.

Polar Coordinates, IV

The primary reason to use polar coordinates is that they will simplify integrals over regions that are portions of circles, because circles have simple descriptions in polar coordinates.

- Specifically, the circle $x^2 + y^2 = a^2$ in rectangular coordinates (over which it is cumbersome to set up double integrals) becomes the much simpler equation $r = a$ in polar coordinates.
- Polar coordinates are also useful in simplifying functions which involve $x^2 + y^2$ or (especially) $\sqrt{x^2 + y^2}$.
- Lines through the origin also have reasonably simple descriptions in polar: the line $y = mx$ becomes the pair of rays $\theta = \tan^{-1}(m)$ and $\theta = \tan^{-1}(m) + \pi$ when written in polar coordinates. (The two rays point in opposite directions.)

Polar Coordinates, V

Example: Describe the following curves in polar coordinates:

1. Find a polar equation for the circle $x^2 + y^2 = 4$.
2. Find a polar equation for $x^2 + y^2 = 4x$.
3. Find a rectangular equation for $\theta = \pi/4$.

Polar Coordinates, V

Example: Describe the following curves in polar coordinates:

1. Find a polar equation for the circle $x^2 + y^2 = 4$.
2. Find a polar equation for $x^2 + y^2 = 4x$.
3. Find a rectangular equation for $\theta = \pi/4$.

- To convert from rectangular to polar we just put in $x = r\cos\theta$ and $y = r\sin\theta$.
- So $x^2 + y^2 = 4$ becomes $r^2\cos^2\theta + r^2\sin^2\theta = 4$, which simplifies to $r^2 = 4$ so that $r = 2$.
- Also, $x^2 + y^2 = 4x$ becomes $r^2\cos^2\theta + r^2\sin^2\theta = 4r\cos\theta$ so that $r^2 = 4r\cos\theta$ so that $r = 4\cos\theta$.
- The polar equation $\theta = \pi/4$ becomes the ray $y = x$, $x > 0$.

Polar Coordinates, VI

To illustrate why we will want to use polar coordinates, consider this double integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

- It is possible to evaluate this integral directly, as written, but it is very messy and would take up this entire slide.
- Notice that the region of integration is defined by $-2 \leq x \leq 2$, $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$, which (if we draw it) we can recognize as the interior of the circle $x^2+y^2=4$.
- If we could set up this integral using polar coordinates, the region would have a much simpler description (it is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$), as would the function (it is just r). We will talk about how to do this next time.

Summary

We discussed double integrals on general regions.

We discussed how to change the order of integration in a double integral.

We quickly reviewed polar coordinates.

Next lecture: Double integrals in polar coordinates.