Math 2321 (Multivariable Calculus)

Lecture #15 of 38 $\sim$ February 24, 2021

Double Integrals

- Double Integrals
- Iterated Integrals

This material represents §3.1.1-3.1.2 from the course notes.

Another Optimization Problem, I

Example: A rectangular trash bin without a top is to be constructed using a total of 3m^2 of sheet metal. Find the maximum possible volume for this storage bin.

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- Suppose the length, width, and height are l meters, w meters, and h meters respectively.
- Then the total surface area is $S = lw + 2lh + 2wh$ square meters and the volume is $V = lwh$ cubic meters.
- We are given $lw + 2lh + 2wh = 3$ and want to maximize lwh.
- We can use Lagrange multipliers with $f(l, w, h) = lwh$ and $g(l, w, h) = lw + 2lh + 2wh$, with constraint $g(l, w, h) = 3$.

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- We can use Lagrange multipliers with $f(l, w, h) = lwh$ and $g(l, w, h) = lw + 2lh + 2wh$, with constraint $g(l, w, h) = 3$.
- We have $\nabla f = \langle wh, lh, lw \rangle$ and $\nabla g = \langle w + 2h, l + 2h, 2l + 2w \rangle$.
- Thus, our system is $wh = \lambda(w + 2h)$, $lh = \lambda(l + 2h)$, $lw = \lambda(2l + 2w)$, and $lw + 2lh + 2wh = 3$.

Another Optimization Problem, II

<u>Example</u>: A rectangular trash bin without a top is to be constructed using a total of 3m^2 of sheet metal. Find the maximum possible volume for this storage bin.

- We have $wh = \lambda(w + 2h)$, $lh = \lambda(l + 2h)$, $lw = \lambda(2l + 2w)$, and $lw + 2lh + 2wh = 3$.
- This system is a bit tricky. What we can do is solve the first three equations for l, w, h in terms of λ and then plug in to the last equation.

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- This system is a bit tricky. What we can do is solve the first three equations for l, w, h in terms of λ and then plug in to the last equation.
- If we divide the first equation by λwh it becomes $\dfrac{1}{\lambda} = \dfrac{1}{h} + \dfrac{2}{w}$.

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- This system is a bit tricky. What we can do is solve the first three equations for l, w, h in terms of λ and then plug in to the last equation.
- If we divide the first equation by λwh it becomes $\dfrac{1}{\lambda} = \dfrac{1}{h} + \dfrac{2}{w}$.
- We can do a similar thing to the other equations to obtain $\dfrac{1}{\lambda} = \dfrac{1}{h} + \dfrac{2}{l}$ and $\dfrac{1}{\lambda} = \dfrac{2}{w} + \dfrac{2}{l}$.

Another Optimization Problem, II

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- This system is a bit tricky. What we can do is solve the first three equations for l, w, h in terms of λ and then plug in to the last equation.
- If we divide the first equation by λwh it becomes $\frac{1}{\lambda} = \frac{1}{h} + \frac{2}{w}$.
- We can do a similar thing to the other equations to obtain $\frac{1}{\lambda} = \frac{1}{h} + \frac{2}{l}$ and $\frac{1}{\lambda} = \frac{2}{w} + \frac{2}{l}$.
- We can then solve for l, w, h (one method: subtract the equations in pairs) to get $h = 2\lambda$ and $l = w = 4\lambda$.

Another Optimization Problem, III

Example: A rectangular trash bin without a top is to be constructed using a total of 3m^2 of sheet metal. Find the maximum possible volume for this storage bin.

- We found $h = 2\lambda$ and $l = w = 4\lambda$.
- Then $lw + 2lh + 2wh = 3$ becomes $16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 3$ so that $\lambda^2 = 1/16$ so $\lambda = \pm 1/4$.
- This yields two points: $(l, w, h) = (1, 1, 1/2)$, $(-1, -1, -1/2)$.
- Since l, w, h are lengths we want the positive solution, and by the physical setup of the problem it must correspond to the maximum volume.
- The desired maximum volume is then $V = lwh = \boxed{\frac{1}{2}\text{m}^3}$.

Overview of §3: Multiple Integration

Now that we've finished multivariable differentiation, we embark on the next major topic: multivariable integration.

- First, we outline the general motivation for double integrals; namely, computing the volume underneath a surface, and briefly mention Riemann sums.
- Next, we explain how to set up and evaluate double integrals, first on rectangles, then on more general regions, along with other topics like changing the order of integration and double integrals in polar coordinates.
- We then discuss the same material for triple integrals, and describe how to do general coordinate changes, along with two 3-dimensional versions of polar coordinates: cylindrical coordinates and spherical coordinates.
- Finally, we do various applications of integration: computing areas, volumes, average values, masses, and moments.

Double Integrals: Motivation, I

Our motivating problem for integration of one variable was to find the area below the curve $y = f(x)$ above an interval on the x-axis.

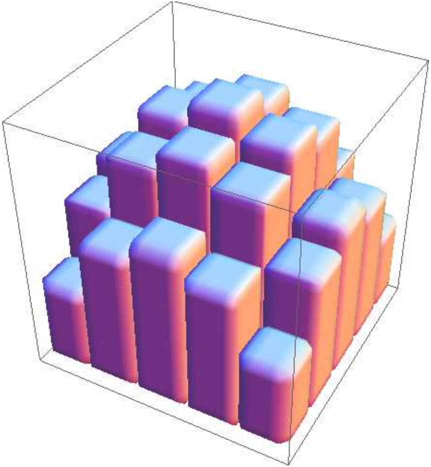
- We now bump this up a dimension: our new goal is to find the volume below the surface $z = f(x, y)$ above a region R in the xy-plane.

The idea is to write down a Riemann sum for the volume in the following way:

- First, we approximate the region R by many small rectangular pieces.
- In each piece, we draw a rectangular prism with base in the xy-plane and upper face intersecting $z = f(x, y)$.
- Then, we take the limit over better and better approximations of the region R, and (so we hope) the collective volume of the rectangular prisms will fill the volume under the graph of $z = f(x, y)$.

Double Integrals: Motivation, II

For $f(x, y) = 2 - x^2 - y^2$, here are some examples of the resulting boxes where R is the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$:



Double Integrals: Motivation, III

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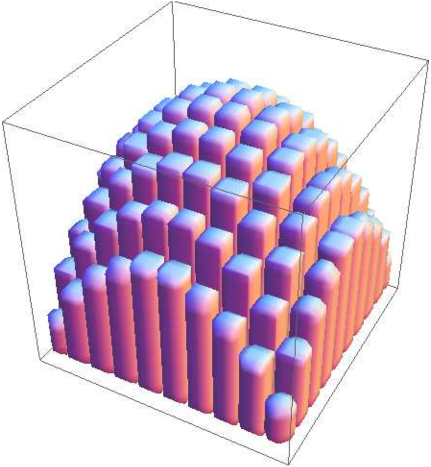
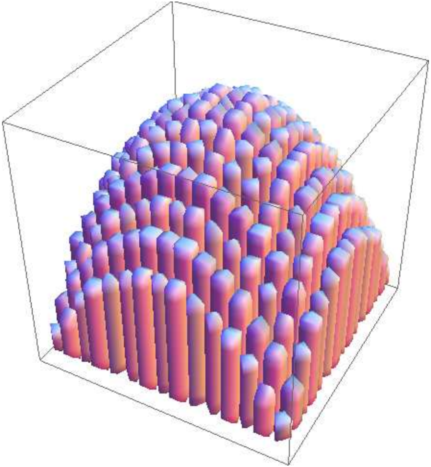


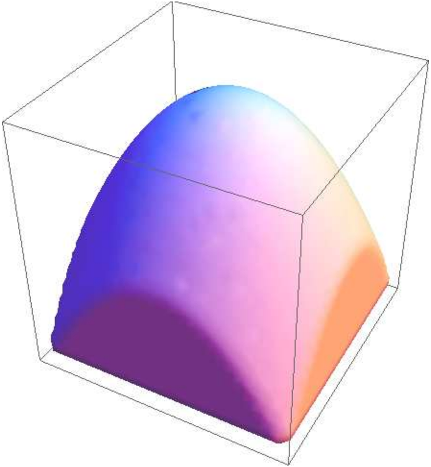
Double Integrals: Motivation, IV

For $f(x, y) = 2 - x^2 - y^2$, here are some examples of the resulting boxes where R is the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$:



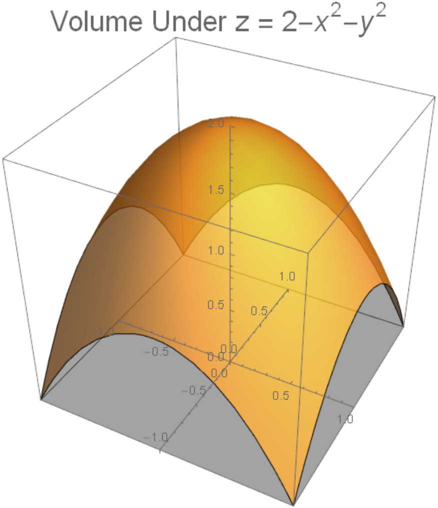
Double Integrals: Motivation, V

Here is the actual solid underneath $f(x, y) = 2 - x^2 - y^2$ where R is the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$:



Double Integrals: Motivation, VI

Here's a more schematic plot of the same solid region:



Riemann Sums, I

We can formalize all of these ideas using Riemann sums. Like with integrals in one dimension, Riemann sums are unwieldy to compute, and we will avoid using this definition.

Definition

For a region R a partition of R into n pieces is a list of disjoint rectangles inside R, where the kth rectangle contains the point (x_k, y_k), has width Δx_k, height Δy_k, and area $\Delta A_k = \Delta y_k \cdot \Delta x_k$.

The norm of the partition P is the largest number among the widths and heights of all of the rectangles in P.

Then, for a continuous function $f(x, y)$ and a partition P of the region R, we define the Riemann sum of $f(x, y)$ on R corresponding to P to be $RS_P(f) = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$.

Riemann Sums, II

The idea now is that we can define the double integral of $f(x, y)$ on R by taking an appropriate limit of Riemann sums:

Definition

For $f(x, y)$ a continuous function, we define the (double) integral of f on the region R, $\iint_R f(x, y)\, dA$, to be the value of L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every partition P with $\mathrm{norm}(P) < \delta$, we have $|RS_P(f) - L| < \epsilon$.

It takes some substantial effort to prove that, if f is continuous, then this limit L actually exists. If you really want to know all the details, you can take Math 4541 (Advanced Calculus), where they prove all of the calculus theorems that I skip.

Riemann Sums, III

The value $\iint\limits_R f(x,y)\,dA$, roughly speaking, is the limit of the Riemann sums of f on R, as the size of the subregions in the partition becomes small.

- Note that our geometric motivation for integration involved finding the area under the graph of a function $z = f(x,y)$.
- As with a function of one variable, however, the definition via Riemann sums does not require that f be nonnegative.
- Accordingly, we interpret the integral of a negative function as giving a negative volume.

Like with integrals of a single variable, double integrals have a number of formal properties that can be deduced from the Riemann sum definition.

Riemann Sums, IV

For C an arbitrary constant and $f(x, y)$ and $g(x, y)$ continuous functions, the following properties hold:

1. Integral of constant: $\iint_R C\, dA = C \cdot \text{Area}(R)$.
2. Constant multiple of a function:
   $\iint_R C\, f(x, y)\, dA = C \cdot \iint_R f(x, y)\, dA$.
3. Addition of functions:
   $\iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA = \iint_R [f(x, y) + g(x, y)]\, dA$.
4. Subtraction of functions:
   $\iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA = \iint_R [f(x, y) - g(x, y)]\, dA$.
5. Nonnegativity: if $f(x, y) \geq 0$, then $\iint_R f(x, y)\, dA \geq 0$.
   - If $f(x, y) \geq g(x, y)$, applying this property to $f - g \geq 0$ shows that $\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$.
6. Union: If R_1 and R_2 don't overlap and have union R, then
   $\iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA = \iint_R f(x, y)\, dA$.

Iterated Integrals, I

You may have noticed that we haven't actually computed any double integrals yet, despite having a perfectly(?) good(ish?) definition as a limit of Riemann sums.

- The reason is that evaluating double integrals via Riemann sums is generally quite hard, even for very simple functions.
- This parallels the situation with partial derivatives: the limit definition, while explicit, is hard to use.
- You might therefore hope that we can play the same game with integrals as we did with derivatives; namely, reduce double integrals down to some kind of single-variable integration problem.
- I will now explain how to do this, using the principle that we can compute volume by integrating cross-sectional area.

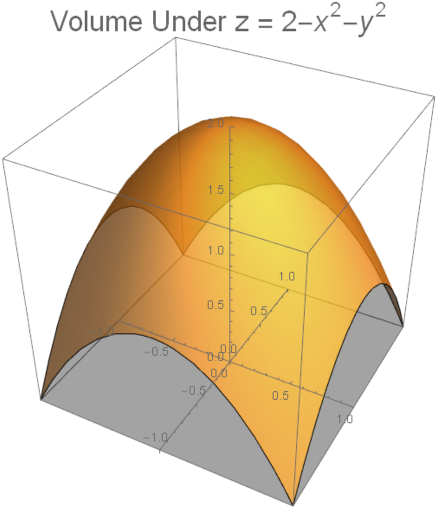
Iterated Integrals, II

To motivate the idea, first suppose the region R is the rectangle with $a \leq x \leq b$, $c \leq y \leq d$, usually written as $[a, b] \times [c, d]$ for short, and our function is $f(x, y)$.

- Imagine taking the solid volume and slicing it into thin pieces perpendicular to the x-axis from $x = a$ to $x = b$.
- Then the volume is given by the integral $\int_a^b A(x)\, dx$, where $A(x)$ is the cross-sectional area at a given x-coordinate.

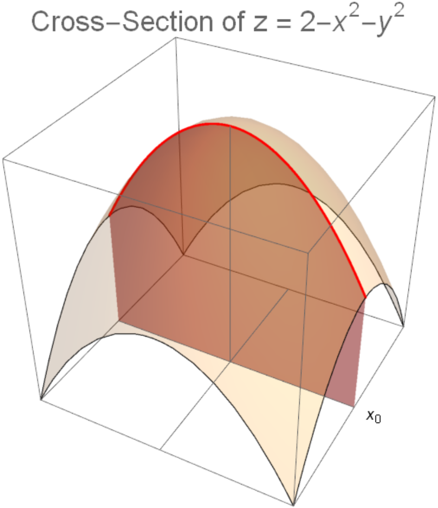
Iterated Integrals, III

Here is a picture of the surface $z = 2 - x^2 - y^2$ from earlier:



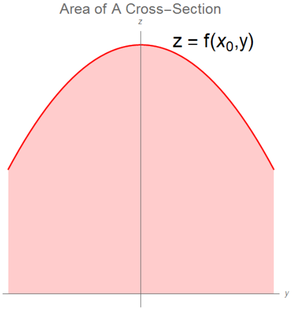
Iterated Integrals, III

Here is a typical cross-section of the surface, taken at $x = x_0$:



Iterated Integrals, IV

Now look at the cross-section:



- The area $A(x_0)$ of this cross-section is simply the area under the curve $z = f(x_0, y)$ between $y = c$ and $y = d$.
- That area is $\int_c^d f(x_0, y)\, dy$, where here we are thinking of x_0 as a constant independent of y.

Iterated Integrals, V

Putting this together shows that the volume under $z = f(x, y)$ above the region $R = [a, b] \times [c, d]$ is given by the iterated integral

$$\int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.$$

- In this integral, we integrate first (on the inside) with respect to the variable y, and then second (on the outside) with respect to the variable x.
- We will usually write iterated integrals without the brackets: $\int_a^b \int_c^d f(x, y)\, dy\, dx$.
- Note that there are two limits of integration, and they are paired with the two variables "inside out": the inner limits $[c, d]$ are paired with the inner differential dy, and the outer limits $[a, b]$ are paired with the outer differential dx.

Iterated Integrals, VI

Example: Calculate $\int_0^1 \int_0^3 x^2 y^2 \, dy \, dx$.

Iterated Integrals, VI

Example: Calculate $\int_0^1 \int_0^3 x^2 y^2 \, dy \, dx$.

- We evaluate the integrals, starting with the inner integral.
- To evaluate the inner integral, we take the antiderivative of $x^2 y^2$ with respect to y, and then evaluate from $y = 0$ to $y = 3$. This gives us a function just of x, which we then integrate from $x = 0$ to $x = 1$:

$$\begin{aligned} \int_0^1 \int_0^3 x^2 y^2 \, dy \, dx &= \int_0^1 \left[\int_0^3 x^2 y^2 \, dy \right] dx = \int_0^1 \left[x^2 \frac{1}{3} y^3 \right] \Big|_{y=0}^{3} dx \\ &= \int_0^1 (9x^2 - 0) \, dx = 3x^3 \Big|_{x=0}^{1} = 3. \end{aligned}$$

Iterated Integrals, VII

Example: Find the volume under the surface $z = 2 - x^2 - y^2$ that lies above the region $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Iterated Integrals, VII

<u>Example</u>: Find the volume under the surface $z = 2 - x^2 - y^2$ that lies above the region $-1 \le x \le 1$, $-1 \le y \le 1$.

- The volume is given by the iterated integral $\int_{-1}^{1} \int_{-1}^{1} \left(2 - x^2 - y^2\right) \, dy \, dx$.

Iterated Integrals, VII

Example: Find the volume under the surface $z = 2 - x^2 - y^2$ that lies above the region $-1 \le x \le 1$, $-1 \le y \le 1$.

- The volume is given by the iterated integral $\int_{-1}^{1} \int_{-1}^{1} \left(2 - x^2 - y^2\right)\, dy\, dx$.
- To evaluate the inner integral $\int_{-1}^{1} \left(2 - x^2 - y^2\right)\, dy$, we view x as constant and take the antiderivative (with respect to y):

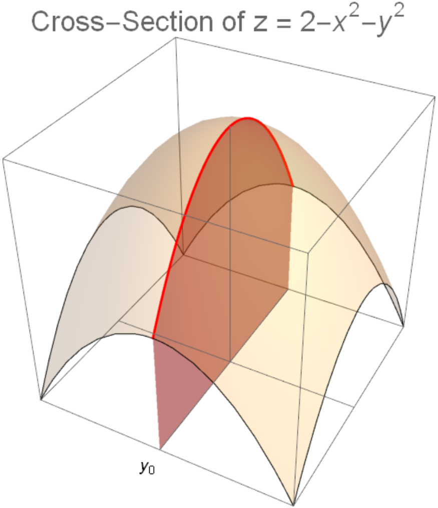
$$\int_{-1}^{1} \left(2 - x^2 - y^2\right)\, dy = \left[2y - x^2 y - \frac{1}{3}y^3\right]\Big|_{y=-1}^{1}$$
$$= \left(2 - x^2 - \frac{1}{3}\right) - \left(-2 + x^2 + \frac{1}{3}\right) = \frac{10}{3} - 2x^2.$$

- Now we can evaluate the outer integral:

$$\int_{-1}^{1} \left(\frac{10}{3} - 2x^2\right)\, dx = \left[\frac{10}{3}x - \frac{2}{3}x^3\right]\Big|_{x=-1}^{1} = \frac{8}{3} - \left(-\frac{8}{3}\right) = \frac{16}{3}.$$

Iterated Integrals, VIII

We could also have sliced perpendicular to the y-axis:



Iterated Integrals, IX

We can go through the same logic as before to compute the volume by slicing perpendicular to the y-axis:

- Explicitly, the volume is the integral $\int_c^d A(y)\,dy$, where $A(y)$ is the cross-sectional area at a given y-coordinate.
- This cross-sectional area is given by the integral $A(y) = \int_a^b f(x,y)\,dx$.
- Thus, the volume should be given by the iterated integral
$$\int_c^d \int_a^b f(x,y)\,dx\,dy.$$
- We can see this integral is in "the other order" (integration order $dx\,dy$ rather than $dy\,dx$) from our previous integral
$$\int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Iterated Integrals, X

Example: Find the volume under the surface $z = 2 - x^2 - y^2$ that lies above the region $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Iterated Integrals, X

Example: Find the volume under the surface $z = 2 - x^2 - y^2$ that lies above the region $-1 \le x \le 1$, $-1 \le y \le 1$.

- The volume is given by $\int_{-1}^{1} \int_{-1}^{1} \left(2 - x^2 - y^2\right)\, dx\, dy$.
- To evaluate the inner integral $\int_{-1}^{1} \left(2 - x^2 - y^2\right)\, dx$, we view y as constant and take the antiderivative (with respect to x):

$$\int_{-1}^{1} \left(2 - x^2 - y^2\right)\, dx = \left[2x - \frac{1}{3}x^3 - xy^2\right]\Big|_{x=-1}^{1}$$
$$= \left(2 - \frac{1}{3} - y^2\right) - \left(-2 + \frac{1}{3} + y^2\right) = \frac{10}{3} - 2y^2.$$

- Now we can evaluate the outer integral:

$$\int_{-1}^{1} \left(\frac{10}{3} - 2y^2\right)\, dy = \left[\frac{10}{3}y - \frac{2}{3}y^3\right]\Big|_{y=-1}^{1} = \frac{8}{3} - \left(-\frac{8}{3}\right) = \frac{16}{3}.$$

Note that we got the same value as before.

Iterated Integrals, XI

We have now three ways to interpret "the integral of a function on the rectangle $[a, b] \times [c, d]$":

1. As an integral $\iint_R f(x, y)\, dA$ via Riemann sums.
2. As an iterated integral $\int_a^b \int_c^d f(x, y)\, dy\, dx$.
3. As an iterated integral $\int_c^d \int_a^b f(x, y)\, dx\, dy$.

We would hope that these definitions all agree. It turns out that as long as the function is continuous on the entire region, they do:

Theorem (Fubini's Theorem)

If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$, then $\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$.

Iterated Integrals, XII

Example: Set up and evaluate $I = \iint_R x^2 y \, dA$, where R is the region $\{(x, y) : 0 \le x \le 2, \ 0 \le y \le 3\}$

1. using the integration order $dy\,dx$.
2. using the integration order $dx\,dy$.

Iterated Integrals, XII

<u>Example</u>: Set up and evaluate $I = \iint_R x^2 y \, dA$, where R is the region $\{(x, y) : 0 \le x \le 2, \ 0 \le y \le 3\}$

1. using the integration order $dy\,dx$.
2. using the integration order $dx\,dy$.

- We've been given the inequalities for the region, so we just need to write down the appropriate iterated integral and then evaluate it.

Iterated Integrals, XIII

Example: Set up and evaluate $I = \iint_R x^2 y \, dA$, where R is the region $\{(x, y) : 0 \le x \le 2, \ 0 \le y \le 3\}$

1. using the integration order $dy \, dx$.

Iterated Integrals, XIII

Example: Set up and evaluate $I = \iint_R x^2 y \, dA$, where R is the region $\{(x, y) : 0 \le x \le 2, \ 0 \le y \le 3\}$

1. using the integration order $dy\, dx$.

- For the order $dy\, dx$, we have $I = \int_0^2 \int_0^3 x^2 y \, dy \, dx$.
- To evaluate the inner integral, we take the antiderivative of $x^2 y$ with respect to y, yielding $\frac{1}{2}x^2 y^2$, and then plug in to evaluate the outer integral. Explicitly:

$$\begin{aligned} I &= \int_0^2 \left[\int_0^3 x^2 y \, dy \right] dx \\ &= \int_0^2 \left[\frac{1}{2} x^2 y^2 \right] \Big|_{y=0}^{3} dx \\ &= \int_0^2 \frac{9}{2} x^2 \, dx = \frac{3}{2} x^3 \Big|_{x=0}^{2} = 12. \end{aligned}$$

Iterated Integrals, XIV

<u>Example</u>: Set up and evaluate $I = \iint_R x^2 y \, dA$, where R is the region $\{(x, y) : 0 \le x \le 2, \ 0 \le y \le 3\}$.

2. using the integration order $dx\,dy$.

Iterated Integrals, XIV

<u>Example</u>: Set up and evaluate $I = \iint_R x^2 y \, dA$, where R is the region $\{(x, y) : 0 \le x \le 2, \; 0 \le y \le 3\}$.

2. using the integration order $dx\,dy$.

- For the order $dx\,dy$ we have $I = \int_0^3 \int_0^2 x^2 y \, dx \, dy$.
- To evaluate the inner integral, we take the antiderivative of $x^2 y$ with respect to x, yielding $\frac{1}{3}x^3 y$, and plug in like before:

$$\begin{aligned} I &= \int_0^3 \left[\int_0^2 x^2 y \, dx \right] dy \\ &= \int_0^3 \left[\frac{1}{3} x^3 y \right] \Big|_{x=0}^{2} dy \\ &= \int_0^3 \frac{8}{3} y \, dy = \frac{4}{3} y^2 \Big|_{y=0}^{3} = 12. \end{aligned}$$

Iterated Integrals, XV

We can also deal with the more general situation where the region R is not a rectangle.

- The principle is very similar to what we were just doing: we will again write down an appropriate iterated integral, but now the inner limits will be functions of the outer variable.
- Once we have actually written down the iterated integral, we can evaluate it just like before.
- On the next slide is an example of the kind of result we will get.

Iterated Integrals, XVI

Example: Calculate $\displaystyle\int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx$.

Iterated Integrals, XVI

<u>Example</u>: Calculate $\int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx$.

- We evaluate the integrals, starting with the inner integral, just like before.
- When we compute the inner integral, the limits will be in terms of x (just plug them in like normal). The result is then a function of x, at which point we evaluate the outer integral:

$$\begin{aligned} \int_0^2 \int_{x^2}^{2x} xy^2 \, dy \, dx &= \int_0^2 \left. \frac{1}{3} xy^3 \right|_{y=x^2}^{2x} dx = \int_0^2 \left[\frac{8}{3} x^4 - \frac{1}{3} x^7 \right] dx \\ &= \left. \left[\frac{8}{15} x^5 - \frac{1}{24} x^8 \right] \right|_{x=0}^{2} = \frac{32}{5}. \end{aligned}$$

Iterated Integrals, XVII

Suppose that the region R is defined by the inequalities $a \le x \le b$, $c(x) \le y \le d(x)$.

- This represents the region above the curve $y = c(x)$ and below the curve $y = d(x)$, between $x = a$ and $x = b$.
- As before, the volume under $z = f(x, y)$ above the region R in the xy-plane is given by the integral $\int_a^b A(x)\, dx$, where $A(x)$ is the cross-sectional area at a given x-coordinate.
- The area $A(x_0)$ of each cross section will be the area under the curve $z = f(x_0, y)$ between $y = c(x_0)$ and $y = d(x_0)$, which is $\int_{c(x_0)}^{d(x_0)} f(x_0, y)\, dy$. (Here, again, we are thinking of x_0 as a constant independent of y.)
- So, the volume is $\int_a^b \left[\int_{c(x)}^{d(x)} f(x, y)\, dy \right] dx$, where now the "inner limits" depend on x.

Iterated Integrals, XVIII

Example: Find the volume under the surface $z = 6 - x^2 - y^2$ that lies above the region $0 \le x \le 1$, $x \le y \le 2x$.

Iterated Integrals, XVIII

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- The volume is $\int_0^1 \int_x^{2x} \left(6 - x^2 - y^2\right)\, dy\, dx$.

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- The volume is $\int_0^1 \int_x^{2x} \left(6 - x^2 - y^2\right)\, dy\, dx$.
- To evaluate the inner integral $\int_x^{2x} \left(6 - x^2 - y^2\right)\, dy$, we take the antiderivative with respect to y:
$$\int_x^{2x} \left(6 - x^2 - y^2\right)\, dy = \left[6y - x^2 y - \frac{1}{3}y^3\right]\Big|_{y=x}^{2x}$$
$$= \left(12x - 2x^3 - \frac{8}{3}x^3\right) - \left(6x - x^3 - \frac{1}{3}x^3\right) = 6x - \frac{10}{3}x^3.$$
- Now we can evaluate the outer integral:
$$\int_0^1 \left(6x - \frac{10}{3}x^3\right)\, dx = \left[3x^2 - \frac{5}{6}x^4\right]\Big|_{x=0}^{1} = \frac{13}{6}.$$

Iterated Integrals, XIX

Just like with rectangular regions, we could also slice using horizontal slices (perpendicular to the y-axis), which will give an iterated integral with integration order $dx\,dy$.

- Explicitly, if the region R is defined by the inequalities $c \le y \le d$, $a(x) \le x \le b(x)$, then the resulting iterated integral will be $\int_c^d \int_{a(x)}^{b(x)} f(x, y)\,dx\,dy$.

Iterated Integrals, XX

Example: Evaluate $\iint_R e^{x+2y}\, dA$, where R is the region $\{(x,y) : 0 \le x \le y,\ 0 \le y \le \ln(3)\}$.

Iterated Integrals, XX

<u>Example</u>: Evaluate $\iint_R e^{x+2y}\,dA$, where R is the region $\{(x,y) : 0 \le x \le y,\ 0 \le y \le \ln(3)\}$.

- Since the x-inequalities depend on y, the order $dx\,dy$ is easiest.
- The resulting iterated integral is $\int_0^{\ln(3)} \int_0^{y} e^{x+2y}\,dx\,dy$.

Iterated Integrals, XX

Example: Evaluate $\iint_R e^{x+2y}\,dA$, where R is the region $\{(x,y) : 0 \le x \le y,\ 0 \le y \le \ln(3)\}$.

- Since the x-inequalities depend on y, the order $dx\,dy$ is easiest.
- The resulting iterated integral is $\int_0^{\ln(3)} \int_0^y e^{x+2y}\,dx\,dy$.
- Now we just evaluate:

$$\begin{aligned}
\int_0^{\ln(3)} \left[\int_0^y e^{x+2y}\,dx\right] dy &= \int_0^{\ln(3)} \left[e^{x+2y}\right]\Big|_{x=0}^{y} dy \\
&= \int_0^{\ln(3)} \left[e^{3y} - e^{2y}\right] dy \\
&= \left(\frac{1}{3}e^{3y} - \frac{1}{2}e^{2y}\right)\Big|_{y=0}^{\ln(3)} = \frac{14}{3}.
\end{aligned}$$

Iterated Integrals, XXI

As with rectangles, we have a version of Fubini's theorem that ensures the value of the integral does not depend on the method we use to compute it:

Theorem (Fubini's Theorem on General Regions)

If $f(x, y)$ is continuous on a region
$R = \{(x, y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ *and*
$R = \{(x, y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$*, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

We will do lots more examples next time.

Summary

We did a few more optimization examples.

We introduced double integrals in rectangular coordinates.

Next lecture: Computing double integrals, order of integration.