Math 2321 (Multivariable Calculus)

Lecture #14 of 38 $\sim$ February 22, 2021

Lagrange Multipliers

- The Method of Lagrange Multipliers
- Constrained Optimization

This material represents §2.6 from the course notes.

Logistics Info

The midterm exams have been graded and the grades, along with grading comments, are posted in Canvas.

- The exams themselves along with their solutions are posted on the course webpage.
- If you have any questions about the exam grading, or if you believe there was some kind of grading error or other issue, please let me know during office hours, via email, or via a private chat message in Zoom.

I have extended this week's WeBWorK by one day, because we won't cover all the material until Wednesday.

Constrained Optimization, I

In the lectures before the exam review, we talked about general optimization problems, of the general form "given a function, maximize it on a region".

- However, many problems are not of this type, but rather of the form "given a function, maximize it subject to some additional constraints".
- Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\ \text{cm}^2$.
- The most natural way to solve such a problem is to eliminate the constraints (i.e., by solving for one of the variables in terms of the others) and then reducing the problem to something without a constraint.
- Then we are left with a regular optimization problem, like the ones we already discussed.

Constrained Optimization, II

<u>Example</u>: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\text{cm}^2$.

- We use the surface area constraint $150\pi\,\text{cm}^2 = 2\pi r^2 + 2\pi rh$ to solve for h in terms of r.
- This gives $h = \frac{150\pi - 2\pi r^2}{2\pi r} = \frac{75 - r^2}{r}$.
- Now we plug in to the volume formula to write it as a function of r alone: this gives $V(r) = \pi r^2 \cdot \frac{75 - r^2}{r} = 75\pi r - \pi r^3$.
- Then $dV/dr = 75\pi - 3\pi r^2$, so by (setting equal to zero) we see the critical points occur for $r = \pm 5$.
- Since we are interested in positive r, we can do a little bit more checking to conclude that the can's volume is indeed maximized at the critical point.
- So the radius is $r = 5\,\text{cm}$, the height is $h = 10\,\text{cm}$, and the resulting maximum volume is $V = 250\pi\text{cm}^3$.

Constrained Optimization, III

This technique works well enough, except that it requires us to solve the constraint equation.

- If instead we had been given even a slightly more complicated constraint, like $r^3 + 2rh + 3h^3 = 200$ (which is quite a bit harder to solve for r or h), we would not have been able to solve the optimization problem.
- What we are seeking instead is a method that does not require us to solve the constraint equation.
- This is what we will discuss now: how to perform a constrained optimization without having to solve the constraint equation (or equations).

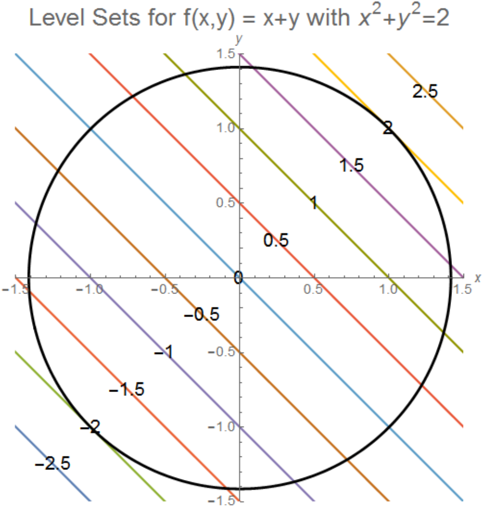
Constrained Optimization, IV

Let's motivate the idea using some geometry:

- Suppose we have a function $f(x, y)$ that we wish to minimize or maximize subject to the constraint $g(x, y) = c$ for some constant c.
- Let's consider some explicit functions: let's look for the minimum and maximum values of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$, which is of the form $g(x, y) = c$ for $g(x, y) = x^2 + y^2$ and $c = 2$.
- Now consider the level sets for the functions f and g.

Constrained Optimization, V

Here are the level sets for $f(x, y) = x + y$ and $x^2 + y^2 = 2$:



Constrained Optimization, VI

Imagine we are walking around the level set $g(x, y) = c$, and consider what the level curves of f are doing as we move around.

- In general the level curves of f will cross the level set $g(x, y) = c$.
- But if we are at a point where f is maximized, then if we walk around nearby that maximum, we will see only level curves of f with a smaller value than the maximum.
- The only way that this can happen is if the level curve of f is tangent to the contour $g(x, y) = c$ at that maximum.
- Since the gradient is orthogonal to (any) tangent curve, this is equivalent to saying that the gradient vector of f is parallel to the gradient vector of g: in other words, that there exists a scalar λ for which $\nabla f = \lambda \nabla g$.

Lagrange Multipliers, I

This observation is the key to the method of Lagrange multipliers, which allows us to solve constrained optimization problems:

Method (Lagrange Multipliers, 2 variables, 1 constraint)

To find the extreme values of $f(x, y)$ subject to a constraint $g(x, y) = c$, as long as $\nabla g \neq \mathbf{0}$, it is sufficient to solve the system of three variables x, y, λ given by $\nabla f = \lambda \nabla g$ and $g(x, y) = c$, and then search among the resulting points (x, y) to find the minimum and maximum.

- The value λ is called a Lagrange multiplier, which is where the name of the procedure comes from.
- If one defines the "Lagrange function" to be $L(x, y, \lambda) = f(x, y) - \lambda \cdot [g(x, y) - c]$, the result above says that the minimum and maximum of $f(x, y)$ subject to $g(x, y) = c$ must occur at critical points of L.

Lagrange Multipliers, II

Proof:

- Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a parametrization of the level curve $g(x, y) = c$ passing through an extreme point of f at $t = 0$.
- Applying the chain rule to f and g yields $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) = \nabla f \cdot \mathbf{r}'(t)$ and $\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x} x'(t) + \frac{\partial g}{\partial y} y'(t) = \nabla g \cdot \mathbf{r}'(t)$.
- Since g is a constant function on the level set $g(x, y) = c$, we have $\partial g / \partial t = 0$ for all t.
- Also, since f has a local extreme point at $t = 0$, we have $\partial f / \partial t = 0$ at $t = 0$.
- So at $t = 0$ we have $\nabla f(0) \cdot \mathbf{r}'(0) = 0$ and $\nabla g(0) \cdot \mathbf{r}'(0) = 0$, so both $\nabla f(0)$ and $\nabla g(0)$ are orthogonal to $\mathbf{r}'(0) \neq \mathbf{0}$.
- Since we are in the plane, $\nabla f(0)$ and $\nabla g(0)$ must be parallel.
- We are also restricted to the curve $g(x, y) = c$, so this equation holds as well.

Lagrange Multipliers, III

Example: Find the minimum and maximum values of $f(x,y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.

Lagrange Multipliers, III

<u>Example</u>: Find the minimum and maximum values of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.

- We use Lagrange multipliers: we have $f(x, y) = x + y$ and $g(x, y) = x^2 + y^2$, so $\nabla f = \langle 1, 1 \rangle$ and $\nabla g = \langle 2x, 2y \rangle$.
- Our system is $\nabla f = \lambda \nabla g$ and $g(x, y) = c$: in other words, $\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle$ and $x^2 + y^2 = 2$.
- Thus we have the system $1 = 2x\lambda$, $1 = 2y\lambda$, and $x^2 + y^2 = 2$.

Lagrange Multipliers, III

<u>Example</u>: Find the minimum and maximum values of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.

- We use Lagrange multipliers: we have $f(x, y) = x + y$ and $g(x, y) = x^2 + y^2$, so $\nabla f = \langle 1,\, 1 \rangle$ and $\nabla g = \langle 2x,\, 2y \rangle$.
- Our system is $\nabla f = \lambda \nabla g$ and $g(x, y) = c$: in other words, $\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle$ and $x^2 + y^2 = 2$.
- Thus we have the system $1 = 2x\lambda$, $1 = 2y\lambda$, and $x^2 + y^2 = 2$.
- Solving the first two equations gives $x = \frac{1}{2\lambda}$ and $y = \frac{1}{2\lambda}$. Then plugging in to the third equation yields $\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 2$, so that $\frac{1}{2\lambda^2} = 2$.
- This yields $\lambda^2 = 1/4$ hence $\lambda = \pm 1/2$.
- Thus, we obtain the two points $(x, y) = (1, 1)$ and $(-1, -1)$.
- Since $f(1, 1) = 2$ and $f(-1, -1) = -2$, the minimum is -2 and the maximum is 2.

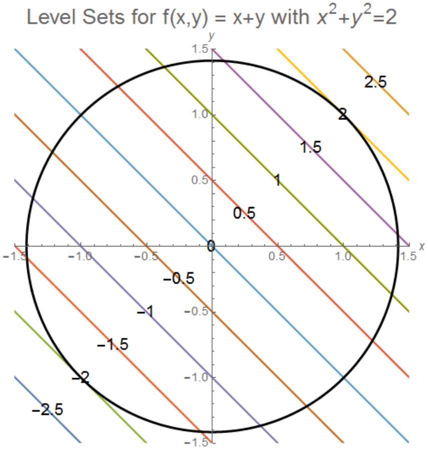
Lagrange Multipliers, IV

Example: Find the minimum and maximum values of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.

Lagrange Multipliers, IV

Example: Find the minimum and maximum values of $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.

- Compare to the picture:



Lagrange Multipliers, V

Example: Find the minimum and maximum values of $f(x, y) = 2x + 3y$ subject to the constraint $x^2 + 4y^2 = 100$.

Lagrange Multipliers, V

Example: Find the minimum and maximum values of $f(x,y) = 2x + 3y$ subject to the constraint $x^2 + 4y^2 = 100$.

- We use Lagrange multipliers: we have $g = x^2 + 4y^2$, so $\nabla f = \langle 2, 3 \rangle$ and $\nabla g = \langle 2x, 8y \rangle$.
- Thus we get $2 = 2x\lambda$, $3 = 8y\lambda$, and $x^2 + 4y^2 = 100$.

Lagrange Multipliers, V

<u>Example</u>: Find the minimum and maximum values of $f(x, y) = 2x + 3y$ subject to the constraint $x^2 + 4y^2 = 100$.

- We use Lagrange multipliers: we have $g = x^2 + 4y^2$, so $\nabla f = \langle 2, 3\rangle$ and $\nabla g = \langle 2x, 8y\rangle$.
- Thus we get $2 = 2x\lambda$, $3 = 8y\lambda$, and $x^2 + 4y^2 = 100$.
- Solving the first two equations gives $x = \dfrac{1}{\lambda}$ and $y = \dfrac{3}{8\lambda}$. Then plugging in to the third equation yields $\left(\dfrac{1}{\lambda}\right)^2 + 4\left(\dfrac{3}{8\lambda}\right)^2 = 100$, so that $\dfrac{1}{\lambda^2} + \dfrac{9}{16\lambda^2} = 100$.
- Multiplying both sides by $16\lambda^2$ yields $25 = 100(16\lambda^2)$, so that $\lambda^2 = 1/64$ hence $\lambda = \pm 1/8$.
- Thus, we obtain the two points $(x, y) = (8, 3)$ and $(-8, -3)$.
- Since $f(8, 3) = 25$ and $f(-8, -3) = -25$, the maximum is $f(8, 3) = 25$ and the minimum is $f(-8, -3) = -25$.

Lagrange Multipliers, VI

Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is 150π cm^2.

Lagrange Multipliers, VI

Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\text{cm}^2$.

- We use Lagrange multipliers.
- We have $f(r,h) = \pi r^2 h$ and $g(r,h) = 2\pi r^2 + 2\pi rh$, so $\nabla f = \left\langle 2\pi rh, \pi r^2 \right\rangle$ and $\nabla g = \langle 4\pi r + 2\pi h, 2\pi r\rangle$.
- This gives the the system $2\pi rh = (4\pi r + 2\pi h)\lambda$, $\pi r^2 = (2\pi r)\lambda$, and $2\pi r^2 + 2\pi rh = 150\pi$.
- Cancelling all of the π factors gives the simpler system $2rh = (4r + 2h)\lambda$, $r^2 = 2r\lambda$, $2r^2 + 2rh = 150$.
- We can also cancel some factors of 2 to get the system $rh = (2r + h)\lambda$, $r^2 = 2r\lambda$, $r^2 + rh = 75$.

Lagrange Multipliers, VII

Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\text{cm}^2$.

- We have $2rh = (4r + 2h)\lambda$, $r^2 = 2r\lambda$, $2r^2 + 2rh = 150$.

Lagrange Multipliers, VII

Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\text{cm}^2$.

- We have $2rh = (4r + 2h)\lambda$, $r^2 = 2r\lambda$, $2r^2 + 2rh = 150$.
- Solving the second equation yields $r = 0$ or $\lambda = r/2$, but $r = 0$ doesn't work in the third equation. So $\lambda = r/2$.
- Plugging into the first equation yields $2rh = (4r + 2h) \cdot r/2$, and cancelling r yields $2h = 2r + h$, so that $h = 2r$.
- Finally, plugging in $h = 2r$ to the third equation yields $2r^2 + 4r^2 = 150$, so that $r^2 = 25$ and $r = \pm 5$.
- The two candidate points are $(r, h) = (5, 10)$ and $(-5, -10)$.
- We only want positive values, so the only point left is $(5, 10)$, which by the physical setup of the problem must be the max.
- Therefore, the maximum volume is $f(5, 10) = 250\pi\text{cm}^3$.

Lagrange Multipliers, VIII

Example: Maximize the volume $V = \pi r^2 h$ of a cylindrical can given that its surface area $SA = 2\pi r^2 + 2\pi rh$ is $150\pi\,\text{cm}^2$.

- The calculations we did here indicate that the cylindrical can with the maximum volume for a fixed surface area has $h = 2r$: in other words, the height equals the diameter.
- Of course, in actual fabrication, the material used to construct the can (i.e., the surface area) is not the only consideration for the shape.
- You can decide for yourself whether most cylindrical containers are actually shaped this way. (Soda cans are generally not: why not?)

Lagrange Multipliers, IX

We can also use Lagrange multipliers as a component of solving optimization-on-a-region problems.

- Specifically, if the boundary of the region can be described as an implicit curve or surface, we can use Lagrange multipliers to identify any potential boundary-critical points.
- This method can often yield easier calculations than the method we discussed last week that required giving a parametrization of the boundary.

Example: Find the absolute minimum and maximum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

Lagrange Multipliers, X

Example: Find the absolute minimum and maximum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- First, we find the critical points: we have $f_x = 2x$ and $f_y = -2y$. Clearly both are zero only at $(x, y) = (0, 0)$, so $(0, 0)$ is the only critical point.
- Next, we analyze the boundary of the region. The boundary is the circle $x^2 + y^2 = 4$, which we can view as the constraint $g(x, y) = 4$ where $g(x, y) = x^2 + y^2$.
- Now we use Lagrange multipliers to analyze the behavior on the boundary: we have $\nabla f = \langle 2x, -2y \rangle$ and $\nabla g = \langle 2x, 2y \rangle$, yielding the system $2x = 2x\lambda$, $-2y = 2y\lambda$, and $x^2 + y^2 = 4$.

Lagrange Multipliers, XI

Example: Find the absolute minimum and maximum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- We have the system $2x = 2x\lambda$, $-2y = 2y\lambda$, and $x^2 + y^2 = 4$.

Lagrange Multipliers, XI

<u>Example</u>: Find the absolute minimum and maximum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- We have the system $2x = 2x\lambda$, $-2y = 2y\lambda$, and $x^2 + y^2 = 4$.
- The first equation yields $x = 0$ or $\lambda = 1$. If $x = 0$ then the third equation yields $y^2 = 4$ so that $y = \pm 2$, and then the second equation is satisfied for $\lambda = -1$: this yields two points $(x, y) = (0, 2)$ and $(0, -2)$.
- Otherwise, if $\lambda = 1$ then the second equation yields $y = 0$, and then the third equation gives $x^2 = 4$ so that $x = \pm 2$: this yields two points $(x, y) = (2, 0)$ and $(-2, 0)$.

Lagrange Multipliers, XI

Example: Find the absolute minimum and maximum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- We have the system $2x = 2x\lambda$, $-2y = 2y\lambda$, and $x^2 + y^2 = 4$.
- The first equation yields $x = 0$ or $\lambda = 1$. If $x = 0$ then the third equation yields $y^2 = 4$ so that $y = \pm 2$, and then the second equation is satisfied for $\lambda = -1$: this yields two points $(x, y) = (0, 2)$ and $(0, -2)$.
- Otherwise, if $\lambda = 1$ then the second equation yields $y = 0$, and then the third equation gives $x^2 = 4$ so that $x = \pm 2$: this yields two points $(x, y) = (2, 0)$ and $(-2, 0)$.
- Our full list of points to analyze is $(0, 0)$, $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$. We have $f(0, 0) = 0$, $f(2, 0) = 4$, $f(0, 2) = -4$, $f(-2, 0) = 4$, and $f(0, -2) = -4$.
- Thus, the maximum is 4 occurring at $(\pm 2, 0)$ and the minimum is -4 occurring at $(0, \pm 2)$.

Lagrange Multipliers, XII

We can also use Lagrange multipliers for systems with more than 2 variables.

Method (Lagrange Multipliers, 3 variables, 1 constraint)

To find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = c$, it is sufficient to solve the system of four variables x, y, z, λ given by $\nabla f = \lambda \nabla g$ and $g(x, y, z) = c$, and then search among the resulting points (x, y, z) to find the minimum and maximum.

The only change is that we have more variables floating around.

The intuition behind the method is the same as before; namely, that the (tangent planes to the) level sets for f and g must be parallel if f has a local extreme point, which means that the gradients must also be parallel.

Lagrange Multipliers, XIII

Example: Find the minimum and maximum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

Lagrange Multipliers, XIII

Example: Find the minimum and maximum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

- We have $g = x^2 + y^2 + z^2$, so $\nabla f = \langle 1, 2, 2 \rangle$ and $\nabla g = \langle 2x, 2y, 2z \rangle$.
- Thus we have the system $1 = 2x\lambda$, $2 = 2y\lambda$, $2 = 2z\lambda$, and $x^2 + y^2 + z^2 = 9$.

Lagrange Multipliers, XIII

Example: Find the minimum and maximum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

- We have $g = x^2 + y^2 + z^2$, so $\nabla f = \langle 1, 2, 2 \rangle$ and $\nabla g = \langle 2x, 2y, 2z \rangle$.
- Thus we have the system $1 = 2x\lambda$, $2 = 2y\lambda$, $2 = 2z\lambda$, and $x^2 + y^2 + z^2 = 9$.
- The first three equations give $x = \frac{1}{2\lambda}$, $y = \frac{1}{\lambda}$, $z = \frac{1}{\lambda}$.
- Then the last equation becomes $[\frac{1}{2\lambda}]^2 + [\frac{1}{\lambda}]^2 + [\frac{1}{\lambda}]^2 = 9$, so that $\frac{9}{4\lambda^2} = 9$ and thus $\lambda = \pm 1/2$.
- This gives two points $(x, y, z) = (1, 2, 2)$ and $(-1, -2, -2)$.
- Since $f(1, 2, 2) = 9$ and $f(-1, -2, -2) = -9$, the maximum is $f(1, 2, 2) = 9$ and the minimum is $f(-1, -2, -2) = -9$.

Lagrange Multipliers, XIV

Example: Find the minimum and maximum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

- You might be interested to know that there actually is a way to solve this problem using vectors, using no calculus at all!
- The idea is to consider vectors $\mathbf{v} = \langle x, y, z \rangle$ and $\mathbf{w} = \langle 1, 2, 2 \rangle$.

Lagrange Multipliers, XIV

Example: Find the minimum and maximum values of $f(x, y, z) = x + 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

- You might be interested to know that there actually is a way to solve this problem using vectors, using no calculus at all!
- The idea is to consider vectors $\mathbf{v} = \langle x, y, z \rangle$ and $\mathbf{w} = \langle 1, 2, 2 \rangle$.
- Then we want to maximize $f = \mathbf{v} \cdot \mathbf{w}$ subject to the condition $||\mathbf{v}||^2 = 9$, which is to say, $||\mathbf{v}|| = 3$.
- But by the dot product theorem, $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| ||\mathbf{w}|| \cos(\theta) = (3)(3)\cos(\theta)$.
- So, the maximum is 9 (with $\cos\theta = 1$) and the minimum is -9 (with $\cos\theta = -1$).

Lagrange Multipliers, XV

Example: Find the absolute minimum and maximum of $f(x, y, z) = x^2 + yz$ on the interior of the ellipsoid $18x^2 + 4y^2 + 9z^2 \leq 72$.

Lagrange Multipliers, XV

Example: Find the absolute minimum and maximum of $f(x,y,z) = x^2 + yz$ on the interior of the ellipsoid $18x^2 + 4y^2 + 9z^2 \leq 72$.

- First, we find the critical points: we have $f_x = 36x$, $f_y = z$, $f_z = y$. Clearly all three are zero only at $(x,y,z) = (0,0,0)$, so (0,0,0) is the only critical point.
- Next, we analyze the boundary of the region. The boundary is the ellipsoid $18x^2 + 4y^2 + 9z^2 = 72$, which we can view as the constraint $g(x,y,z) = 72$ for $g(x,y,z) = 18x^2 + 4y^2 + 9z^2$.
- Now we use Lagrange multipliers: we have $\nabla f = \langle 2x, z, y \rangle$ and $\nabla g = \langle 36x, 8y, 18z \rangle$, yielding the system $2x = 36x\lambda$, $z = 8y\lambda$, $y = 18z\lambda$, and $x^2 + 4y^2 + 9z^2 = 72$.

Lagrange Multipliers, XVI

Example: Find the absolute minimum and maximum of $f(x, y, z) = x^2 + yz$ on the interior of the ellipsoid $18x^2 + 4y^2 + 9z^2 \leq 72$.

Lagrange Multipliers, XVI

Example: Find the absolute minimum and maximum of $f(x, y, z) = x^2 + yz$ on the interior of the ellipsoid $18x^2 + 4y^2 + 9z^2 \leq 72$.

- We have $2x = 36x\lambda$, $z = 8y\lambda$, $y = 18z\lambda$, $18x^2 + 4y^2 + 9z^2 = 72$.
- Plugging the second equation into the third yields $y = 144\lambda^2 y$ so that $y = 0$ or $\lambda^2 = 1/144$.
- If $y = 0$ then the second equation would give $z = 0$, and then the fourth equation would become $18x^2 = 72$ so that $x = \pm 2$. We obtain two points: $(x, y, z) = (2, 0, 0)$ and $(-2, 0, 0)$.
- Otherwise, $y \neq 0$ and so $\lambda^2 = 1/144$. Then, because $\lambda \neq 1$, the first equation requires $x = 0$. The system then reduces to $z = \pm\frac{2}{3}y$ and $4y^2 + 9z^2 = 72$, so plugging in yields $4y^2 + 4y^2 = 72$ and thus $y^2 = 9$ so that $y = \pm 3$.
- We obtain four points: $(x, y, z) = (0, 3, 2)$, $(0, 3, -2)$, $(0, -3, 2)$, and $(0, -3, -2)$.

Lagrange Multipliers, XVII

Example: Find the absolute minimum and maximum of $f(x, y, z) = x^2 + yz$ on the interior of the ellipsoid $18x^2 + 4y^2 + 9z^2 \leq 72$.

- Our full list of points to analyze is $(0, 0, 0)$, $(2, 0, 0)$, $(-2, 0, 0)$, $(0, 3, 2)$, $(0, 3, -2)$, $(0, -3, 2)$, and $(0, -3, -2)$.
- We have $f(0, 0, 0) = 0$, $f(\pm 2, 0, 0) = 4$, $f(0, 3, 2) = f(0, -3, -2) = 6$, $f(0, 3, -2) = f(0, -3, 2) = -6$.
- So, the maximum is 6, occurring at $(0, 3, 2)$ and $(0, -3, -2)$, while the minimum is -6, occurring at $(0, 3, -2)$ and $(0, -3, 2)$.

Lagrange Multipliers, XVIII [FOR FUN ONLY]

For completeness we also mention that there is an analogous procedure for a problem with two constraints:

Method (Lagrange Multipliers, 3 variables, 2 constraints)

To find the extreme values of $f(x, y, z)$ *subject to a pair of constraints* $g(x, y, z) = c$ *and* $h(x, y, z) = d$*, it is sufficient to solve the system of five variables* x, y, z, λ, μ *given by* $\nabla f = \lambda \nabla g + \mu \nabla h$*,* $g(x, y, z) = c$*, and* $h(x, y, z) = d$*, and then search among the resulting triples* (x, y, z) *to find the minimum and maximum.*

The method also works with more than three variables, and has a natural generalization to more than two constraints. (It is fairly rare to encounter systems with more than two constraints.)

We won't do an example of this method since such problems tend to be very long.

Summary

We discussed constrained optimization and the method of Lagrange multipliers.

We discussed a number of examples of Lagrange multipliers problems.

Next lecture: Double integrals.