Math 2321 (Multivariable Calculus)

Lecture #13 of 38 $\sim$ February 18, 2021

Midterm #1 Review #2

Midterm 1 Exam Topics

The topics for the exam are as follows:

- 3D graphing, level sets
- Vectors, vector operations
- Dot and cross products
- Lines and planes in 3-space
- Curves and motion in 3-space (including **T** and **N**)
- Partial derivatives
- Directional derivatives and the gradient
- Tangent lines and planes
- Linearization
- The chain rule, implicit differentiation
- Critical points and their classification

This represents §1.1 – 2.5.1 from the notes and WeBWorKs 1-4.

Exam Information

I have sent emails (through Canvas) to confirm your testing window. Please verify that it is correct.

Any questions about exam logistics?

Review Problems, I

(#1e) Suppose $\mathbf{v} = \langle 3, 0, -4 \rangle$ and $\mathbf{w} = \langle -1, 6, 2 \rangle$. Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

Review Problems, I

(#1e) Suppose $\mathbf{v} = \langle 3, 0, -4 \rangle$ and $\mathbf{w} = \langle -1, 6, 2 \rangle$. Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

- Recall that the dot product theorem says $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos(\theta)$.
- Therefore, the angle is $\theta = \cos^{-1}(\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \, ||\mathbf{w}||}) = \boxed{\cos^{-1}\left[\frac{-11}{5\sqrt{41}}\right]} \approx 1.9215$ radians.

Review Problems, II

(#4e) A particle has position $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t) \rangle$ at time t. Find the unit tangent vector $\mathbf{T}(t)$.

Review Problems, II

(#4e) A particle has position $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$ at time t. Find the unit tangent vector $\mathbf{T}(t)$.

- The unit tangent vector, by definition, is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$.
- Since $\mathbf{r}'(t) = \langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle$, we have $||\mathbf{r}'(t)|| = \sqrt{(-3\sin t)^2 + (5\cos t)^2 + (-4\sin t)^2} = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} = \sqrt{25\sin^2 t + 25\cos^2 t} = 5$.
- Thus, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \boxed{\left\langle -\dfrac{3}{5}\sin(t), \cos(t), -\dfrac{4}{5}\sin(t)\right\rangle}$.

Review Problems, III

(#7b.f) Let $f(x, y, z) = x^3yz^2$. Find the minimum and maximum rates of change of f at the point $(1, 2, 1)$, and the unit vector directions in which the minimum and maximum rates occur.

Review Problems, III

(#7b.f) Let $f(x, y, z) = x^3yz^2$. Find the minimum and maximum rates of change of f at the point $(1, 2, 1)$, and the unit vector directions in which the minimum and maximum rates occur.

- The maximum rate is in the direction of ∇f and the magnitude is $||\nabla f||$, while the minimum rate is the opposite direction with the opposite sign.
- As $\nabla f = \langle 3x^2yz^2, x^3z^2, 2x^3yz \rangle$, $\nabla f(1, 2, 1) = \langle 6, 1, 4 \rangle$.
- So the maximum rate is $||\nabla f(1, 2, 1)|| = ||\langle 6, 1, 4 \rangle|| = \boxed{\sqrt{53}}$ in the direction of $\frac{\nabla f}{||\nabla f||} = \boxed{\frac{1}{\sqrt{53}} \langle 6, 1, 4 \rangle}$.
- The minimum rate is $-||\nabla f(1, 2, 1)|| = \boxed{-\sqrt{53}}$ in the direction of $-\frac{\nabla f}{||\nabla f||} = \boxed{-\frac{1}{\sqrt{53}} \langle 6, 1, 4 \rangle}$.

Review Problems, IV

(#11b) Find and classify the critical points for $f(x, y) = x^4 + y^2 - 8x^2 + 4y$.

Review Problems, IV

(#11b) Find and classify the critical points for $f(x, y) = x^4 + y^2 - 8x^2 + 4y$.

- We set $f_x = 0$ and $f_y = 0$ to find the critical points, then use the second derivatives test to classify them.
- We have $f_x = 4x^3 - 16x$ and $f_y = 2y + 4$.
- Solving $f_x = 0$ gives $4x^3 - 16x = 0$ so $4x(x^2 - 4) = 0$, meaning that $x = -2, 0, 2$.
- Likewise, $f_y = 0$ gives $2y + 4 = 0$ so that $y = -2$.
- We get 3 critical points: $(x, y) = \boxed{(-2, -2), (0, -2), (2, -2)}$.
- We also compute $f_{xx} = 12x^2 - 16$, $f_{xy} = 0$, $f_{yy} = 2$, so that $D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 16)(2) - 0^2$.
- At $(-2, -2)$ and $(2, -2)$, $D = 64 > 0$ and $f_{xx} = 32 > 0$ so $\boxed{(-2, -2) \text{ and } (2, -2) \text{ are local minima}}$.
- At $(0, -2)$, $D = -32$ so $\boxed{(0, -2) \text{ is a saddle point}}$.

Review Problems, V

(#2g) Find an equation for the plane containing the vectors $\langle 1, 2, -1 \rangle$ and $\langle 2, -1, 1 \rangle$ and the point $(1, -1, 2)$.

Review Problems, V

(#2g) Find an equation for the plane containing the vectors $\langle 1, 2, -1\rangle$ and $\langle 2, -1, 1\rangle$ and the point $(1, -1, 2)$.

- The normal vector is orthogonal to the two given vectors, so it is given by their cross product.
- Thus, the normal vector is
$$\langle 1, 2, -1\rangle \times \langle 2, -1, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \langle 1, -3, -5\rangle.$$
- Thus, the plane's equation is $x - 3y - 5z = d$ for some constant d.
- Since the plane passes through $(1, -1, 2)$, plugging in yields $d = -6$.
- So the equation is $\boxed{x - 3y - 5z = -6}$.

Review Problems, VI

(#12a) Find the min and max of $f(x, y) = x^2 - 2xy + 3y^2 - 4y$ on the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$.

Review Problems, VI

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- First, critical points: $f_x = 2x - 2y$, $f_y = -2x + 6y - 4$ so $y = x$ so $4x - 4 = 0$ so $(x,y) = (1,1)$. Now the boundary:

1. $(0,0)$ to $(2,0)$: parametrized by $(x,y) = (2t, 0)$, $0 \le t \le 1$. Then $f = 4t^2$, $f' = 8t$, so f' is zero at $t = 0$. Yields boundary-crit point $(0,0)$ and endpoints $(0,0)$, $(2,0)$.
2. $(2,0)$ to $(2,4)$: parametrized by $(x,y) = (2, 4t)$, $0 \le t \le 1$. Then $f = 48t^2 - 32t + 4$, $f' = 96t - 32$, zero at $t = 1/3$. Yields $(2, 4/3)$ and endpoints $(2,0)$, $(2,4)$.
3. $(0,0)$ to $(2,4)$: parametrized by $(x,y) = (2t, 4t)$, $0 \le t \le 1$. Then $f = 36t^2 - 16t$, $f' = 72t - 16$, zero at $t = 2/9$. Yields $(4/9, 8/9)$ and endpoints $(0,0)$, $(2,4)$.

Point list is $(1,1)$, $(0,0)$, $(2,0)$, $(2,4/3)$, $(2,4)$, $(4/9,8/9)$. Minimum is -2 at $(1,1)$, maximum is 20 at $(2,4)$.

Review Problems, VII

(#9a) Suppose $\frac{\partial f}{\partial x}(1,5) = 9$, $\frac{\partial f}{\partial y}(1,5) = -3$, $\frac{\partial f}{\partial x}(2,-2) = 4$, and $\frac{\partial f}{\partial y}(2,-2) = 5$, where $x(s,t)$ and $y(s,t)$ are such that $x(1,5) = 2$, $y(1,5) = -2$, $\frac{\partial x}{\partial s}(1,5) = 3$, $\frac{\partial x}{\partial t}(1,5) = 2$, $\frac{\partial y}{\partial s}(1,5) = 4$, and $\frac{\partial y}{\partial t}(1,5) = -2$. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (1,5)$.

Review Problems, VII

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- We apply the chain rule.
- If $s = 1$ and $t = 5$ then $x = 2$ and $y = -2$. This means we want the partial derivatives of f that are evaluated at $(x,y) = (2,-2)$ rather than $(x,y) = (1,5)$.
- Here, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = 4 \cdot 3 + 5 \cdot 4 = \boxed{32}$.

Review Problems, VIII

(#9b) Suppose $\dfrac{\partial f}{\partial x}(1,5) = 9$, $\dfrac{\partial f}{\partial y}(1,5) = -3$, $\dfrac{\partial f}{\partial x}(2,-2) = 4$, and $\dfrac{\partial f}{\partial y}(2,-2) = 5$, where $x(s,t)$ and $y(s,t)$ are such that $x(1,5) = 2$, $y(1,5) = -2$, $\dfrac{\partial x}{\partial s}(1,5) = 3$, $\dfrac{\partial x}{\partial t}(1,5) = 2$, $\dfrac{\partial y}{\partial s}(1,5) = 4$, and $\dfrac{\partial y}{\partial t}(1,5) = -2$. Find $\dfrac{\partial f}{\partial t}$ at $(s,t) = (1,5)$.

Review Problems, VIII

(#9b) Suppose $\frac{\partial f}{\partial x}(1,5) = 9$, $\frac{\partial f}{\partial y}(1,5) = -3$, $\frac{\partial f}{\partial x}(2,-2) = 4$, and $\frac{\partial f}{\partial y}(2,-2) = 5$, where $x(s,t)$ and $y(s,t)$ are such that $x(1,5) = 2$, $y(1,5) = -2$, $\frac{\partial x}{\partial s}(1,5) = 3$, $\frac{\partial x}{\partial t}(1,5) = 2$, $\frac{\partial y}{\partial s}(1,5) = 4$, and $\frac{\partial y}{\partial t}(1,5) = -2$. Find $\frac{\partial f}{\partial t}$ at $(s,t) = (1,5)$.

- Here, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = 4 \cdot 2 + 5 \cdot (-2) = \boxed{-2}$.

Review Problems, IX

(#6ab) A potato is fired into the air at time $t = 0$s from the origin in a vacuum with initial velocity $\mathbf{v}(0) = (4\mathbf{i} + 8\mathbf{j} + 80\mathbf{k})\,\mathrm{m/s}$. Assuming that the only force acting on the potato is the downward acceleration due to gravity of $\mathbf{a}(t) = -10\mathbf{k}\,\mathrm{m/s}^2$, find the velocity and position of the potato at time t seconds.

Review Problems, IX

(#6ab) A potato is fired into the air at time $t = 0\text{s}$ from the origin in a vacuum with initial velocity $\mathbf{v}(0) = (4\mathbf{i} + 8\mathbf{j} + 80\mathbf{k})\,\text{m/s}$. Assuming that the only force acting on the potato is the downward acceleration due to gravity of $\mathbf{a}(t) = -10\mathbf{k}\,\text{m/s}^2$, find the velocity and position of the potato at time t seconds.

- Since $\mathbf{v}'(t) = \mathbf{a}(t)$, taking the antiderivative of $\mathbf{a}(t)$ gives $\mathbf{v}(t)$.
- This yields $\mathbf{v}(t) = \langle C_1, C_2, C_3 - 10t \rangle$ m/s.
- Plugging in the initial condition $\mathbf{v}(0) = \langle 4, 8, 80 \rangle$ m/s gives $C_1 = 4,\ C_2 = 8,\ C_3 = 80$.
- Thus, $\mathbf{v}(t) = \boxed{\langle 4,\ 8,\ 80 - 10t \rangle\,\text{m/s}}$.
- In the same way, by integrating $\mathbf{v}(t)$ and plugging in the initial condition, we obtain $\mathbf{r}(t) = \boxed{\left\langle 4t,\ 8t,\ 80t - 5t^2 \right\rangle\,\text{m}}$.

Review Problems, X

(#6cd) A potato is fired at time $t = 0\,\text{s}$ and at time $t\,\text{s}$, it has position $\mathbf{r}(t) = \left\langle 4t, 8t, 80t - 5t^2 \right\rangle$ m and velocity $\mathbf{v}(t) = \langle 4, 8, 80 - 10t \rangle$ m/s. Find the total time that the potato is in the air, and the potato's speed when it hits the ground.

Review Problems, X

(#6cd) A potato is fired at time $t = 0\,\text{s}$ and at time $t\,\text{s}$, it has position $\mathbf{r}(t) = \left\langle 4t,\, 8t,\, 80t - 5t^2 \right\rangle$ m and velocity $\mathbf{v}(t) = \langle 4,\, 8,\, 80 - 10t \rangle$ m/s. Find the total time that the potato is in the air, and the potato's speed when it hits the ground.

- The potato hits the ground when its height (i.e., its z-coordinate is $0\,\text{m}$).
- This occurs when $80t - 5t^2 = 0$ so that $t = 0\,\text{s}$ or $t = 16\,\text{s}$. Since it is fired at $t = 0\,\text{s}$, it hits the ground at time $t = \boxed{16\,\text{s}}$.
- The speed is then $||\mathbf{r}'(16\,\text{s})|| = ||\langle 4, 8, -80 \rangle\,\text{m/s}|| = \sqrt{4^2 + 8^2 + (-80)^2}\,\text{m/s} = \boxed{\sqrt{6480}\,\text{m/s}}$.

Review Problems, XI

(#10f) Suppose $f(x, y) = x^3 + 3xy$. Find an equation for the tangent plane to the graph of $z = f(x, y)$ at $(x, y) = (-1, 1)$.

Review Problems, XI

(#10f) Suppose $f(x,y) = x^3 + 3xy$. Find an equation for the tangent plane to the graph of $z = f(x,y)$ at $(x,y) = (-1,1)$.

- We rewrite the equation as $x^3 + 3xy - z = 0$ and then use the fact that for an implicit surface of the form $g(x,y,z) = 0$, the gradient $\nabla g(P)$ is the normal vector to the tangent plane to the surface at a point P.
- If $x = -1$ and $y = 1$, then $z = f(-1,1) = -4$. So our point P is $(-1,1,-4)$.
- For $g(x,y,z) = x^3 + 3xy - z$, $\nabla g = \left\langle 3x^2 + 3y, 3x, -1 \right\rangle$.
- So, $\nabla g(-1,1,-4) = \langle 6, -3, -1 \rangle$.
- Therefore, the tangent plane has equation $6x - 3y - z = d$ for some constant d.
- Plugging in $P = (-1,1,-4)$ shows $d = -5$, so we get the equation $\boxed{6x - 3y - z = -5}$.

Review Problems, XII

(#10b) Suppose $f(x, y) = x^3 + 3xy$. Find the rate of change of f at $(x, y) = (1, 2)$ in the direction toward the origin.

Review Problems, XII

(#10b) Suppose $f(x,y) = x^3 + 3xy$. Find the rate of change of f at $(x,y) = (1,2)$ in the direction toward the origin.

- This is asking for a directional derivative.
- Remember that the directional derivative of f in the direction of the unit vector $\mathbf{v}$ is $\nabla f(P) \cdot \mathbf{v}$.
- We have $\nabla f = \left\langle 3x^2 + 3y, 3x \right\rangle$ so $\nabla f(1,2) = \langle 9, 3 \rangle$.
- The vector towards the origin from $(1,2)$ is $\langle -1, -2 \rangle$, which has length $\sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$.
- Thus, the unit vector towards the origin from $(1,2)$ is $\frac{1}{\sqrt{5}} \langle -1, -2 \rangle$.
- So, the rate of change is $\langle 9, 3 \rangle \cdot \frac{1}{\sqrt{5}} \langle -1, -2 \rangle = -\frac{15}{\sqrt{5}} = \boxed{-3\sqrt{5}}$.

Review Problems, XIII

(#2a) Find an equation for the plane parallel to $x + 2y - 3z = 1$ containing the point $(2, -1, 2)$.

Review Problems, XIII

(#2a) Find an equation for the plane parallel to $x + 2y - 3z = 1$ containing the point $(2, -1, 2)$.

- Parallel planes have the same normal vector.
- So, the normal vector for the desired plane is $\langle 1, 2, -3 \rangle$, meaning its equation is $x + 2y - 3z = d$ for some d.
- Plugging in $(2, -1, 2)$ shows that $d = -6$, so the equation is $\boxed{x + 2y - 3z = -6}$.

Review Problems, XIV

(#11d) Find and classify the critical points for $f(x, y) = x^3 - 3xy + 3y^2$.

Review Problems, XIV

(#11d) Find and classify the critical points for $f(x, y) = x^3 - 3xy + 3y^2$.

- We have $f_x = 3x^2 - 3y$ and $f_y = -3x + 6y$, so we get the system $3x^2 - 3y = 0$ and $-3x + 6y = 0$.
- Solving the second equation for x gives $x = 2y$.
- Then plugging into the first equation yields $12y^2 - 3y = 0$, so that $y = 0$ or $y = 1/4$.
- If $y = 0$ then $x = 2y = 0$ while if $y = 1/4$ then $x = 2y = 1/2$.
- So there are two critical points: $(x, y) = \boxed{(0, 0),\ (1/2, 1/4)}$.
- We have $f_{xx} = 6x$, $f_{xy} = -3$, $f_{yy} = 6$, so Then $D = (6x)(6) - (-3)^2$.
- At $(0, 0)$, $D = -9 < 0$, so $\boxed{(0, 0) \text{ is a saddle point}}$.
- At $(1/2, 1/4)$, $D = 9 > 0$ and $f_{xx} = 3 > 0$, so $\boxed{(1/2, 1/4) \text{ is a local minimum}}$.

Review Problems, XV

(#7c.g) Find the linearization to $g(x, y, z) = \ln(x^2 + y^2 + z^2)$ at the point $(x, y, z) = (2, 1, 1)$.

Review Problems, XV

(#7c.g) Find the linearization to $g(x,y,z) = \ln(x^2+y^2+z^2)$ at the point $(x,y,z) = (2,1,1)$.

- The linearization is $L(x,y,z) = g(2,1,1)$ $+ g_x(2,1,1)(x-2) + g_y(2,1,1)(y-1) + g_z(2,1,1)(z-1)$.
- We have $g_x = 2x/(x^2+y^2+z^2)$, $g_y = 2y/(x^2+y^2+z^2)$, and $g_z = 2z/(x^2+y^2+z^2)$.
- So $g(2,1,1) = \ln 6$, $g_x(2,1,1) = 2/3$, $g_y(2,1,1) = 1/3$, and $g_z(2,1,1) = 1/3$.
- Thus, $L(x,y,z) = \boxed{\ln(6) + \frac{2}{3}(x-2) + \frac{1}{3}(y-1) + \frac{1}{3}(z-1)}$.

Review Problems, XVI

(#8b.x) Suppose that z is defined implicitly as a function of x and y by the relation $e^{x-yz} + xz = 9$. Find $\dfrac{\partial z}{\partial x}$.

Review Problems, XVI

(#8b.x) Suppose that z is defined implicitly as a function of x and y by the relation $e^{x-yz} + xz = 9$. Find $\frac{\partial z}{\partial x}$.

- By implicit differentiation, we have $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \boxed{-\frac{e^{x-yz} + z}{-ye^{x-yz} + x}}$.

Review Problems, XVII

(#1d) If $\mathbf{w} = \langle -1, 6, 2 \rangle$, find a vector of length 4 in the same direction as $\mathbf{w}$.

Review Problems, XVII

(#1d) If $\mathbf{w} = \langle -1, 6, 2 \rangle$, find a vector of length 4 in the same direction as $\mathbf{w}$.

- Note that $||\mathbf{w}|| = \sqrt{(-1)^2 + 6^2 + 2^2} = \sqrt{41}$.
- A unit vector in the same direction as $\mathbf{w}$ is $\dfrac{\mathbf{w}}{||\mathbf{w}||}$.
- Scaling this vector by 4 yields the desired vector: explicitly, it is $4\dfrac{\mathbf{w}}{||\mathbf{w}||} = \boxed{\left\langle \dfrac{-4}{\sqrt{41}}, \dfrac{24}{\sqrt{41}}, \dfrac{8}{\sqrt{41}} \right\rangle}$.

Review Problems, XVIII

(#2c) Find a parametrization for the line parallel to $\langle x, y, z \rangle = \langle 1 - 2t,\ 3 + 2t,\ 2 + 5t \rangle$ containing the point $(1, 1, 1)$.

Review Problems, XVIII

(#2c) Find a parametrization for the line parallel to $\langle x, y, z\rangle = \langle 1 - 2t,\ 3 + 2t,\ 2 + 5t\rangle$ containing the point $(1, 1, 1)$.

- We need the direction vector for the new line. We can find it by noting that parallel lines have the same direction vector.
- Since the direction vector for the given line is $\mathbf{v} = \langle -2, 2, 5\rangle$, that is also the direction vector for the new line.
- Since the new line passes through $P = (1, 1, 1)$, its parametrization is given by $\langle x, y, z\rangle = P + t\mathbf{v} = \langle 1, 1, 1\rangle + t\langle -2, 2, 5\rangle$.
- Explicitly, this is $\boxed{\langle x, y, z\rangle = \langle 1 - 2t, 1 + 2t, 1 + 5t\rangle}$.

Review Problems, XIX

(#10c) Find the unit vector direction in which $f(x, y) = x^3 + 3xy$ is decreasing the fastest at $(x, y) = (2, 0)$.

Review Problems, XIX

(#10c) Find the unit vector direction in which $f(x, y) = x^3 + 3xy$ is decreasing the fastest at $(x, y) = (2, 0)$.

- This direction is the unit vector pointing in the opposite direction of the gradient vector $\nabla f(P)$.
- Since $\nabla f = \langle 3x^2 + 3y, 3x\rangle$ we have $\nabla f(P) = \langle 12, 6\rangle$, with length $||\nabla f(P)|| = \sqrt{12^2 + 6^2} = 6\sqrt{5}$.
- The desired unit vector is then given by $-\dfrac{\nabla f(P)}{||\nabla f(P)||} = \boxed{\left\langle -\dfrac{2}{\sqrt{5}}, -\dfrac{1}{\sqrt{5}} \right\rangle}$.

Review Problems, XX

(#1b) Find the minimum and maximum values of $f(x, y) = x + y$ on the circular region $x^2 + y^2 \leq 4$.

Review Problems, XX

(#1b) Find the minimum and maximum values of $f(x, y) = x + y$ on the circular region $x^2 + y^2 \leq 4$.

- First, f has no critical points, since $f_x = 1$ and $f_y = 1$.
- For the boundary, we could parametrize it using $x = 2\cos t$, $y = 2\sin t$ for $0 \leq t \leq 2\pi$.
- Then $f = 2\cos t + 2\sin t$, so $f' = -2\sin t + 2\cos t$ which is zero when $\sin t = \cos t$, yielding $t = \pi/4$, $5\pi/4$ so $(x, y) = (\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$.
- So, we see that the min is $-2\sqrt{2}$ at $(-\sqrt{2}, -\sqrt{2})$ and the max is $2\sqrt{2}$ at $(\sqrt{2}, \sqrt{2})$.

Review Problems, XXI

(#2f) Find a parametrization for the intersection of the planes $x + y + 2z = 4$ and $2x - y - z = 5$.

Review Problems, XXI

(#2f) Find a parametrization for the intersection of the planes $x + y + 2z = 4$ and $2x - y - z = 5$.

- We need the normal vector (which is perpendicular to the direction vectors of the two lines) and a point the plane passes through (which we can find by trial and error).
- The direction vector is orthogonal to $\langle 1, 1, 2\rangle$ and $\langle 2, -1, -1\rangle$, so is given by cross product $\langle 1, 1, 2\rangle \times \langle 2, -1, 1\rangle = \langle 1, 5, -3\rangle$.
- To find a point in the plane we try picking a value for one coordinate.
- Setting $z = 0$ gives $x + y = 4$ and $2x - y = 5$ which eventually yields $x = 3$, $y = 1$: thus a point in both planes is $(3, 1, 0)$.
- We get a parametrization $\boxed{\langle x, y, z\rangle = \langle 3 + t,\ 1 + 5t,\ -3t\rangle}$.

Review Problems, XXII

(#7a.g) Find the rate of change of $g(x, y, z) = \ln(x^2 + y^2 + z^2)$ in the direction of $\mathbf{v} = \langle 2, -1, 2 \rangle$ at the point $(1, 1, 1)$.

Review Problems, XXII

(#7a.g) Find the rate of change of $g(x, y, z) = \ln(x^2 + y^2 + z^2)$ in the direction of $\mathbf{v} = \langle 2, -1, 2 \rangle$ at the point $(1, 1, 1)$.

- The directional derivative is the dot product $\nabla g(P) \cdot \mathbf{w}$ of the gradient $\nabla g(P)$ with the unit direction vector $\mathbf{w}$.
- Note that $\nabla g = \frac{1}{x^2+y^2+z^2} \langle 2x, 2y, 2z \rangle$.
- So, $\nabla g(1, 1, 1) = \frac{1}{3} \langle 2, 2, 2 \rangle$.
- Also, $||\mathbf{v}|| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$.
- So the unit direction vector is $\mathbf{w} = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \dfrac{1}{3}\langle 2, -1, 2 \rangle$.
- The directional derivative is thus
  $\nabla g(P) \cdot \mathbf{w} = \dfrac{1}{3} \langle 2, 2, 2 \rangle \cdot \dfrac{1}{3} \langle 2, -1, 2 \rangle = \boxed{\dfrac{2}{3}}$.

Review Problems, XXIII

(#4f) A particle has position $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$ at time t. Find the unit normal vector $\mathbf{N}(t)$.

(Earlier we found $\mathbf{T}(t) = \left\langle -\frac{3}{5}\sin(t), \cos(t), -\frac{4}{5}\sin(t)\right\rangle$.)

Review Problems, XXIII

(#4f) A particle has position $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$ at time t. Find the unit normal vector $\mathbf{N}(t)$.

(Earlier we found $\mathbf{T}(t) = \left\langle -\frac{3}{5}\sin(t), \cos(t), -\frac{4}{5}\sin(t)\right\rangle$.)

- The unit normal vector, by definition, is $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$.
- Since $\mathbf{T}'(t) = \left\langle -\frac{3}{5}\cos(t), -\sin(t), -\frac{4}{5}\cos(t)\right\rangle$, we have $||\mathbf{T}'(t)|| = \sqrt{(9/25)\cos^2 t + \sin^2 t + (16/25)\cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = 1$.
- So $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \boxed{\left\langle -\frac{3}{5}\cos(t), -\sin(t), -\frac{4}{5}\cos(t)\right\rangle}$.

Summary

We did more review problems for midterm 1.

Next lecture: Lagrange multipliers.