Math 2321 (Multivariable Calculus)

Lecture #12 of 38 $\sim$ February 17, 2021

Midterm #1 Review #1

Midterm 1 Exam Topics

The topics for the exam are as follows:

- 3D graphing, level sets
- Vectors, vector operations, dot and cross products
- Lines, planes, curves, and motion in 3-space (includes **T**, **N**)
- Partial derivatives
- Directional derivatives and gradients
- Tangent lines and planes
- Linearization
- The chain rule, implicit differentiation
- Critical points and classification (minima, maxima, saddles)
- Optimization on a region

This represents §1.1 – 2.5.2 from the notes and WeBWorKs 1-4.

Exam Information, I

Some notes on the exam format:

- The exam has the same format as if it were being given in class, in person.
- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- There are approximately 6 pages of material, about 1/5 multiple choice and the rest free response.
- I have set up a Piazza poll for you to select your desired exam window. Please make your selection by **6pm today**. I will confirm your selection via Canvas notification.

Exam Information, II

Some notes on the exam format:

- The "official" exam time limit is 65 minutes. I give you 25 extra minutes of working time, so you really have 90 minutes. To this are added 30 minutes of turnaround time (for downloading, printing, scanning, and uploading).
- Thus, the exam windows are 120 minutes in length (180 if you have a DRC-approved testing-time accommodation).
- Your Canvas assignment submission will disappear after your exam window finishes. After that time, you cannot submit via Canvas. Emailed exams may be subject to point penalties.

Exam Information, III

Some notes on the exam format:

- You are allowed to use calculators / equivalent computing technology on the exam.
- You are also allowed to use notes. Normally this would not be allowed, however it does not seem reasonable to disallow notes when you are taking the exam remotely.
- However, you must show all relevant details and identify whenever you used a calculator to do a calculation. You are expected to justify all calculations and show all work. Correct answers without appropriate work may not receive full credit.

Exam Information, IV

Some notes on the exam format:

- If you have any questions during the exam (e.g., you believe there is an error, or you want to ask for clarification), please email me.
- It may not be possible for me to respond. If you do not receive a response, please do your best to solve the problem anyway. Any actual errors on the exam will be resolved in your favor, as long as this is reasonable.
- Collaboration of any other kind is not allowed. You may not discuss anything about the exam with anyone other than me (the instructor) until 5pm Eastern on Tuesday, February 23rd. This includes Piazza posts.

Exam Information, V

Any questions about exam logistics?

So as to make today's review more interesting, I will do problems in roughly random order from the review sheet.

Review Problems, I

(#2b) Find a parametrization for the line containing $(2, -1, 4)$ and $(3, 6, 2)$.

Review Problems, I

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- The direction vector is $\langle 3, 6, 2\rangle - \langle 2, -1, 4\rangle = \langle 1, 7, -2\rangle$.
- Thus, a parametrization is $\boxed{\langle x, y, z\rangle = \langle 2 + t,\ -1 + 7t,\ 4 - 2t\rangle}$.
- It is also possible to start with the other point, which gives a different parametrization, or to orient the direction vector differently.

Review Problems, II

(#1c) Suppose $\mathbf{v} = \langle 3, 0, -4 \rangle$. Find a unit vector in the opposite direction of $\mathbf{v}$.

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- The unit vector in the direction of $\mathbf{v}$ is $\dfrac{\mathbf{v}}{||\mathbf{v}||}$, so the unit vector in the opposite direction is $-\dfrac{\mathbf{v}}{||\mathbf{v}||}$
- We have $||\mathbf{v}|| = \sqrt{3^2 + 0^2 + (-4)^2} = \sqrt{25} = 5$.
- Thus, the desired vector is $\dfrac{-\mathbf{v}}{||\mathbf{v}||} = \boxed{\left\langle -\dfrac{3}{5}, 0, \dfrac{4}{5} \right\rangle}$.

Review Problems, III

(#7a.f) Let $f(x, y, z) = x^3yz^2$. Find the rate of change of f in the direction of $\mathbf{v} = \langle 2, -1, 2 \rangle$ at the point $(1, 1, 1)$.

Review Problems, III

(#7a.f) Let $f(x, y, z) = x^3yz^2$. Find the rate of change of f in the direction of $\mathbf{v} = \langle 2, -1, 2 \rangle$ at the point $(1, 1, 1)$.

- The directional derivative is the dot product $\nabla f(P) \cdot \mathbf{w}$ of the gradient $\nabla f(P)$ with the unit direction vector $\mathbf{w}$.
- Note that $\nabla f = \left\langle 3x^2yz^2,\ x^3z^2,\ 2x^3yz \right\rangle$.
- So, $\nabla f(P) = \langle 3, 1, 2 \rangle$.
- Also, $||\mathbf{v}|| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$.
- So the unit direction vector is $\mathbf{w} = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \dfrac{1}{3}\langle 2, -1, 2 \rangle$.
- The directional derivative is $\langle 3, 1, 2 \rangle \cdot \frac{1}{3} \langle 2, -1, 2 \rangle = \boxed{3}$.

Review Problems, IV

(#11a) Find and classify the critical points for
$f(x, y) = xy - x^2 - y^2 - 2x - 2y$.

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$f(x, y) = xy - x^2 - y^2 - 2x - 2y$.

- First we solve $f_x = f_y = 0$ to identify critical points, and then use the second derivatives test to classify them.
- We have $f_x = y - 2x - 2$ and $f_y = x - 2y - 2$.
- So $y - 2x - 2 = 0$ and $x - 2y - 2 = 0$.
- The first equation gives $y = 2x + 2$ and then the second equation gives $x - 2(2x + 2) - 2 = 0$ so that $-3x - 6 = 0$ so $x = -2$. Then $y = -2$ also.
- So we get a single critical point $(x, y) = \boxed{(-2, -2)}$.
- We have $f_{xx} = -2$, $f_{xy} = 1$, $f_{yy} = -2$.
- Thus, $D = (-2)(-2) - 1^2 = 3$. Since $D > 0$ and $f_{xx} < 0$, there is a $\boxed{\text{local maximum at } (-2, -2)}$.

Review Problems, V

(#2e) Find an equation for the plane containing the points $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 3, 5)$.

Review Problems, V

(#2e) Find an equation for the plane containing the points $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 3, 5)$.

- We need the normal vector to the plane, which we can get by taking the cross product of two vectors in the plane, and we can get those by taking differences of the given points.
- We can take vectors $\langle 2, 1, 2\rangle - \langle 1, 0, 1\rangle = \langle 1, 1, 1\rangle$ and $\langle 3, 3, 5\rangle - \langle 1, 0, 1\rangle = \langle 2, 3, 4\rangle$.
- Then the normal vector is given by the cross product $\langle 1, 1, 1\rangle \times \langle 2, 3, 4\rangle = \langle 1, -2, 1\rangle$.
- Then the equation is $x - 2y + z = d$. Plugging in $(1, 0, 1)$ then shows we need $d = 2$, so the equation is $\boxed{x - 2y + z = 2}$.
- If we did the cross product in the other order, we would instead get the equation $\boxed{-x + 2y - z = -2}$, which is the same plane, just written differently.

Review Problems, VI

(#10e) Suppose $f(x, y) = x^3 + 3xy$. Find the linearization of f at $(x, y) = (1, 3)$ and use it to estimate $f(1.01, 3.02)$.

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- For this function of two variables, the linearization is $L(x, y) = f(1, 3) + f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3)$.
- We have $f_x = 3x^2 + 3y$ and $f_y = 3x$, so $f(1, 3) = 10$, $f_x(1, 3) = 12$, and $f_y(1, 3) = 3$.
- Therefore $L(x, y) = f(1, 3) + f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3) = \boxed{10 + 12(x - 1) + 3(y - 3)}$.
- The approximation from the linearization is $f(1.01, 3.02) \approx L(1.01, 3.02) = 10 + 12(0.01) + 3(0.02) = \boxed{10.18}$.

Review Problems, VII

(#4abcd) At time t hours, a particle has position
$\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$ kilometers.
Find the particle's velocity, acceleration, speed, and the distance travelled by the particle between $t = 0$h and $t = 1$h.

Review Problems, VII

(#4abcd) At time t hours, a particle has position
$\mathbf{r}(t) = \langle 3\cos(t),\, 5\sin(t),\, 4\cos(t)\rangle$ kilometers.
Find the particle's velocity, acceleration, speed, and the distance travelled by the particle between $t = 0$h and $t = 1$h.

- The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \boxed{\langle -3\sin(t),\, 5\cos(t),\, -4\sin(t)\rangle}$ kilometers per hour.
- The acceleration is
  $\mathbf{v}'(t) = \mathbf{r}''(t) = \boxed{\langle -3\cos(t),\, -5\sin(t),\, -4\cos(t)\rangle}$ kilometers per hour squared.
- The speed is $||\mathbf{v}(t)|| = \sqrt{9\sin^2(t) + 25\cos^2(t) + 16\sin^2(t)}$
  $= \sqrt{25} = \boxed{5}$ kilometers per hour.
- The arclength is $s = \int_0^1 ||\mathbf{v}(t)||\, dt = \int_0^1 5\, dt = \boxed{5}$ kilometers.

Review Problems, VIII

(#8b.y) Assuming that z is defined implicitly as a function of x and y by $e^{x-yz} + xz = 9$, find $\dfrac{\partial z}{\partial y}$.

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- By implicit differentiation, $\dfrac{\partial z}{\partial y} = -\dfrac{f_y}{f_z} = \boxed{-\dfrac{-ze^{x-yz}}{-ye^{x-yz} + x}}$.

Review Problems, IX

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(#10g) Find and classify the critical points for $f(x, y) = x^3 + 3xy$.

- We have $f_x = 3x^2 + 3y$ and $f_y = 3x$.
- Thus we get $3x^2 + 3y = 0$ and $3x = 0$.
- The second equation requires $x = 0$, and then the first equation gives $y = 0$.
- So there is one critical point: $(x, y) = \boxed{(0, 0)}$.
- We have $D = f_{xx}f_{yy} - (f_{xy})^2 = (6x)(0) - 3^2 = -9$.
- Thus, since $D(0, 0) = -9 < 0$, that means $\boxed{(0, 0) \text{ is a saddle point}}$.

Review Problems, X

(#12c) Find the minimum and maximum values of $f(x, y) = 2x^2 - y$ on the region between $y = 8x$ and $y = x^2$.

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- First, $f_x = 4x$ and $f_y = -1$ so there are no critical points.
- The boundary has two components:
  1. Segment on $y = 8x$ from $(0, 0)$ to $(8, 64)$, parametrized by $x = 8t$, $y = 64t$ for $0 \le t \le 1$. Then $f = 128t^2 - 64t$, so $f' = 256t - 64$, zero at $t = 1/4$. Yields $(2, 16)$, endpoints $(0, 0)$, $(8, 64)$.
  2. Component along $y = x^2$, parametrized by $x = t$, $y = t^2$ for $0 \le t \le 8$. Then $f = t^2$, so $f' = 2t$, zero at $t = 0$. Yields $(0, 0)$, endpoints $(0, 0)$, $(8, 64)$.
- Point list is $(0, 0)$, $(8, 64)$, $(2, 16)$.
- Minimum is $\boxed{-8 \text{ at } (2, 16)}$, maximum is $\boxed{64 \text{ at } (8, 64)}$.

Review Problems, XI

(#8a) Find an equation for the tangent plane to the surface $e^{x-yz} + xz = 9$ at $(x, y, z) = (4, 2, 2)$.

Review Problems, XI

(#8a) Find an equation for the tangent plane to the surface $e^{x-yz} + xz = 9$ at $(x, y, z) = (4, 2, 2)$.

- The surface is of the form $f(x, y, z) = 9$ where $f(x, y, z) = e^{x-yz} + 3yz$.
- Then the gradient $\nabla f(P)$ is the normal vector to the tangent plane at P.
- We compute $\nabla f = \langle e^{x-yz} + z, \; -ze^{x-yz}, \; -ye^{x-yz} + x \rangle$, so $\nabla f(4, 2, 2) = \langle 3, -2, 2 \rangle$.
- Thus, the tangent plane has equation $3x - 2y + 2z = d$. Plugging in $(4, 2, 2)$ shows that $d = 12$, so the equation is $\boxed{3x - 2y + 2z = 12}$.

Review Problems, XII

(#7b.g) Let $g(x, y, z) = \ln(x^2 + y^2 + z^2)$.
Find the minimum and maximum rates of change of g at the point $(1, 2, 1)$, and the unit vector directions in which the minimum and maximum rates of change occur.

Review Problems, XII

(#7b.g) Let $g(x, y, z) = \ln(x^2 + y^2 + z^2)$.
Find the minimum and maximum rates of change of g at the point $(1, 2, 1)$, and the unit vector directions in which the minimum and maximum rates of change occur.

- The maximum rate is in the direction of ∇g and the magnitude is $||\nabla g||$, while the minimum rate is the opposite direction with the opposite sign.
- As $\nabla g = \frac{1}{x^2+y^2+z^2}\langle 2x, 2y, 2z\rangle$, $\nabla g(1, 2, 1) = \frac{1}{6}\langle 2, 4, 2\rangle$.
- So the maximum rate is $||\nabla g(1, 2, 1)|| = \left|\left|\frac{1}{6}\langle 2, 4, 2\rangle\right|\right| = \boxed{\frac{1}{6}\sqrt{24}}$ in the direction of $\frac{\nabla g}{||\nabla g||} = \boxed{\frac{1}{\sqrt{24}}\langle 2, 4, 2\rangle}$.
- The minimum rate is $-||\nabla g(1, 2, 1)|| = \boxed{-\frac{1}{6}\sqrt{24}}$ in the direction of $-\frac{\nabla g}{||\nabla g||} = \boxed{-\frac{1}{\sqrt{24}}\langle 2, 4, 2\rangle}$.

Summary

We discussed exam logistics.

We did some review problems.

Next lecture: Midterm 1 review (part 2)