Math 2321 (Multivariable Calculus)

Lecture #11 of 38 $\sim$ February 11, 2021

Optimization on a Region

- Applied Optimization
- Optimization on a Region

This material represents §2.5.1-2.5.2 from the course notes.
[This is the last new material ahead of Midterm 1. Next week's lectures will be devoted to exam review.]

Recall

Definition

A <u>*critical point*</u> *of a function* $f(x,y)$ *is a point where* ∇f *is zero or undefined. Equivalently,* (x_0,y_0) *is a critical point if* $f_x(x_0,y_0)=0=f_y(x_0,y_0)$*, or either* $f_x(x_0,y_0)$ *or* $f_y(x_0,y_0)$ *is undefined.*

We have a similar definition for functions of more than two variables.

A local minimum or maximum of a function of several variables can only occur at a critical point: otherwise, the function would increase in the direction of ∇f and decrease in the direction of $-\nabla f$.

Optimization, I

By searching for critical points, we can solve optimization problems that require us to find the global minimum or maximum of a function on its domain.

- Note, however, that merely identifying critical points is not the end of the discussion.
- Specifically, if the function's domain is unbounded, the value of the function could attain values lower than any local minimum, or values higher than any local maximum.
- This is even true in the one-variable case: $f(x) = x^3 - 3x$ has a unique local min $x = 1$ and a unique local max $x = -1$ but it takes arbitrary large positive and negative values.
- Therefore, we need to examine what happens to f as any (or all) of the individual variables tend to ∞.
- In particular, if $f \to \infty$ as any of the variables tends to ∞, then f must have a global minimum.

Optimization, II

<u>Example</u>: Find the minimum value of the function $f(a, b) = (b - 1)^2 + (a + b - 2)^2 + (4a + b - 9)^2$.

Optimization, II

Example: Find the minimum value of the function $f(a,b) = (b-1)^2 + (a+b-2)^2 + (4a+b-9)^2$.

- Note that if a or b is large, then at least one of the terms $(b-1)^2$, $(a+b-2)^2$ must be large. Therefore, f must have a global minimum value.
- Then $f_a = 2(a+b-2) + 8(4a+b-9) = 34a + 10b - 76$ and $f_b = 2(b-1) + 2(a+b-2) + 2(4a+b-9) = 10a + 6b - 24$.
- Thus, $34a + 10b - 76 = 0$ and $10a + 6b - 24 = 0$.
- Solving the second equation for b yields $b = 4 - \frac{5}{3}a$, and then the first equation yields $34a + 10(4 - \frac{5}{3}a) - 76 = 0$ so that $\frac{52}{3}a - 36 = 0$ and thus $a = 27/13$ and $b = 7/13$.
- Since there is one critical point, it must be the global min.
- The minimum value of f is thus $f(27/13, 7/13) = 8/13$.

Optimization on a Region, I

We often want to find the absolute minimum or maximum of a function $f(x, y)$ on a finite region, rather than the whole plane.

- In this situation, we must also analyze the function's behavior on the boundary of the region, because the boundary could contain the minimum or maximum.
- For example, the extreme values of $f(x, y) = x^2 - y^2$ on the square $0 \le x \le 1$, $0 \le y \le 1$ occur at two of the "corner points": the minimum is -1 occurring at $(0, 1)$, and the maximum $+1$ occurring at $(1, 0)$.
- We can see that these two points are actually the minimum and maximum on this region without calculus: since squares of real numbers are always nonnegative, on the region in question we have $-1 \le -y^2 \le x^2 - y^2 \le x^2 \le 1$.

The boundary must be included in the region (i.e., the region must be closed and bounded) in order for these extreme values to exist.

Optimization on a Region, II

Unfortunately, unlike the case of a function of one variable where the boundary of an interval $[a, b]$ is very simple (namely, the two values $x = a$ and $x = b$), the boundary of a region in the plane or in higher-dimensional space can be rather complicated.

- One approach is to find a parametrization $(x(t), y(t))$ of the boundary of the region. (This may require breaking the boundary into several pieces, depending on the shape of the region.)
- Then, by plugging the parametrization of the boundary curve into the function, we obtain a function $f(x(t), y(t))$ of the single variable t, which we can then analyze to determine the behavior of the function on the boundary.
- Another approach, which we will discuss in a week and a half, is to use Lagrange multipliers.

Optimization on a Region, III

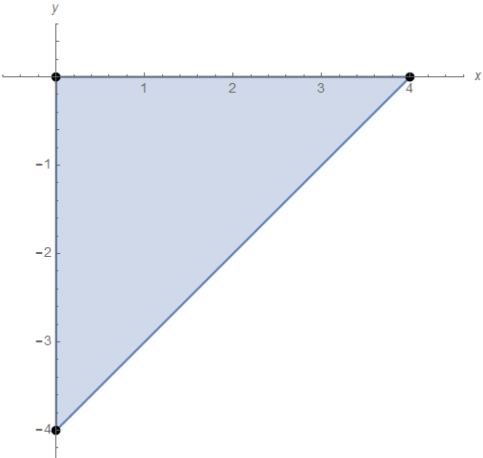
Here is the procedure for finding the minimum and maximum values of a function $f(x, y)$ on a given closed, bounded region R:

0. Draw the region.
1. Find all of the critical points of f that lie inside the region R.
2. Parametrize the boundary of the region R (separating into several components if necessary) as $x = x(t)$ and $y = y(t)$, then plug in the parametrization to obtain a function of t, $f(x(t), y(t))$. Then search for "boundary-critical" points, where the t-derivative $\frac{d}{dt}$ of $f(x(t), y(t))$ is zero. Also include endpoints, if the boundary components have them.
3. Plug the full list of critical and boundary-critical points into f, and find the largest and smallest values.

You should make a clear list of all the candidate points when you solve one of these problems.

Optimization on a Region, IV

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:



Optimization on a Region, V

<u>Example</u>: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- First, we find the critical points.

Optimization on a Region, V

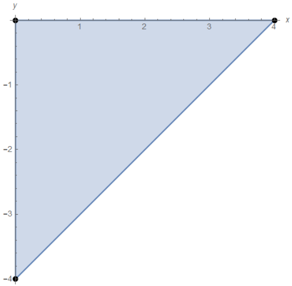
<u>Example</u>: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- First, we find the critical points.
- We have $f_x = 3x^2 + 6y$ and $f_y = -3y^2 + 6x$.
- Solving $f_y = 0$ yields $x = y^2/2$ and then plugging into $f_x = 0$ gives $y^4/4 + 2y = 0$ so that $y(y^3 + 8) = 0$.
- Therefore, $y = 0$ or $y = -2$.
- If $y = 0$ then $x = y^2/2 = 0$ so we get $(0, 0)$.
- If $y = -2$ then $x = y^2/2 = 2$ so we get $(2, -2)$.
- Overall we get the critical points $(0, 0)$ and $(2, -2)$.

Optimization on a Region, VI

<u>Example</u>: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Now, we analyze the boundary of the region:



The boundary has 3 pieces:

1. The line segment from $(0, 0)$ to $(4, 0)$.
2. The line segment from $(0, -4)$ to $(4, 0)$.
3. The line segment from $(0, 0)$ to $(0, -4)$.

- We can parametrize the line segment from (a, b) to (c, d) as $x = a + (c - a)t$, $y = b + (d - b)t$ for $0 \le t \le 1$.

Optimization on a Region, VII

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Component #1 is the line segment joining $(0, 0)$ to $(4, 0)$.
- This component is parametrized by $x = 4t$, $y = 0$ for $0 \leq t \leq 1$.

Optimization on a Region, VII

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Component #1 is the line segment joining $(0, 0)$ to $(4, 0)$.
- This component is parametrized by $x = 4t$, $y = 0$ for $0 \leq t \leq 1$.
- On this component, we have $f(t, 0) = 64t^3$, which has a critical point only at $t = 0$, which corresponds to $(x, y) = (0, 0)$.
- We must also include the two endpoints of the segment, which are $(0, 0)$ and $(4, 0)$.

Optimization on a Region, VIII

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Component #2 is the line segment joining $(0, -4)$ to $(4, 0)$.
- This component is parametrized by $x = 4t$, $y = 4t - 4$ for $0 \leq t \leq 1$.

Optimization on a Region, VIII

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Component #2 is the line segment joining $(0, -4)$ to $(4, 0)$.
- This component is parametrized by $x = 4t$, $y = 4t - 4$ for $0 \leq t \leq 1$.
- On this component, we have $f(4t, 4t - 4) = 32(9t^2 - 9t + 2)$. To identify critical points we take the derivative, yielding $32(18t - 9)$, so there is a critical point for $t = 1/2$, corresponding to $(x, y) = (2, -2)$.
- We must also include the two endpoints of the segment, which are $(4, 0)$ and $(0, -4)$.

Optimization on a Region, IX

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Component #3 is the line segment joining $(0, 0)$ to $(0, -4)$.
- This component is parametrized by $x = 0$, $y = -4t$ for $0 \leq t \leq 1$.

Optimization on a Region, IX

Example: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Component #3 is the line segment joining $(0, 0)$ to $(0, -4)$.
- This component is parametrized by $x = 0$, $y = -4t$ for $0 \le t \le 1$.
- On this component, we have $f(0, -4t) = 64t^3$, which has a critical point at $t = 0$, corresponding to $(x, y) = (0, 0)$.
- We must also include the two endpoints of the segment, which are $(0, 0)$ and $(0, -4)$.

Optimization on a Region, X

<u>Example</u>: Find the absolute minimum and maximum of $f(x, y) = x^3 + 6xy - y^3$ on the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, -4)$:

- Putting all of the analysis together yields a list of four points to analyze: $(0, 0)$, $(4, 0)$, $(0, -4)$, and $(2, -2)$.
- Now we plug in to see that $f(0, 0) = 0$, $f(4, 0) = 64$, $f(0, -4) = 64$, $f(2, -2) = -8$.
- Therefore, the maximum value of f is 64, occurring at $(4, 0)$ and $(0, -4)$, and the minimum value is -8, occurring at $(2, -2)$.

Optimization on a Region, XI

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

Optimization on a Region, XI

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- First, we find the critical points.
- We have $f_x = 2x$ and $f_y = -2y$. Clearly both functions are zero only at $(x, y) = (0, 0)$, so $(0, 0)$ is the only critical point.
- Now we must analyze the boundary of the region.

Optimization on a Region, XI

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- First, we find the critical points.
- We have $f_x = 2x$ and $f_y = -2y$. Clearly both functions are zero only at $(x, y) = (0, 0)$, so $(0, 0)$ is the only critical point.
- Now we must analyze the boundary of the region.
- The boundary is the circle $x^2 + y^2 = 4$, which is a single curve that we can parametrize by $x = 2\cos(t)$, $y = 2\sin(t)$ for $0 \leq t \leq 2\pi$.

Optimization on a Region, XI

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

Optimization on a Region, XI

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- The boundary is parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$.

Optimization on a Region, XI

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- The boundary is parametrized by $x = 2\cos(t)$, $y = 2\sin(t)$.
- Then $f(t) = f(2\cos(t), 2\sin(t)) = 4\cos^2 t - 4\sin^2 t$.
- Taking the derivative yields $\dfrac{df}{dt} =$ $4\left[2\cos(t) \cdot (-\sin(t)) - 2\sin(t) \cdot \cos(t)\right] = -16\cos(t) \cdot \sin(t)$.
- The derivative is equal to zero when $\cos(t) = 0$ or when $\sin(t) = 0$.
- For $0 \leq t \leq 2\pi$ this gives the possible values $t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$, yielding the points $(x, y) = (2, 0)$, $(0, 2)$, $(-2, 0)$, $(0, -2)$, and $(2, 0)$ [again].

Optimization on a Region, XII

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

Optimization on a Region, XII

Example: Find the absolute maximum and minimum of $f(x, y) = x^2 - y^2$ on the closed disc $x^2 + y^2 \leq 4$.

- Our full list of points to analyze is $(0, 0)$, $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$.
- We have $f(0, 0) = 0$, $f(2, 0) = 4$, $f(0, 2) = -4$, $f(-2, 0) = 4$, and $f(0, -2) = -4$.
- Therefore, the maximum value is 4, occurring at $(2, 0)$ and $(-2, 0)$, while the minimum value is -4, occurring at $(0, 2)$ and $(0, -2)$.

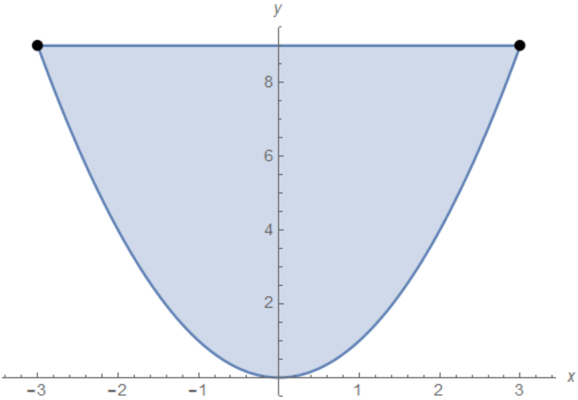
Optimization on a Region, XIII

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

Optimization on a Region, XIII

Example: Find the absolute maximum and minimum of $f(x,y) = xy - 3x$ on the region with $x^2 \le y \le 9$.

- Here is the region:



Optimization on a Region, XIV

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

- First, we find the critical points.

Optimization on a Region, XIV

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

- First, we find the critical points.
- Since $f_x = y - 3$ and $f_y = x$, there is a single critical point $(0, 3)$.
- Next, we analyze the boundary of the region, which (as the plot on the previous slide indicates) has 2 components:
    1. The line segment from $(-3, 9)$ to $(3, 9)$.
    2. The parabolic arc along $y = x^2$ from $(-3, 9)$ to $(3, 9)$.

Optimization on a Region, XV

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

- Component #1, the line segment from $(-3, 9)$ to $(3, 9)$.

Optimization on a Region, XV

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \le y \le 9$.

- Component #1, the line segment from $(-3, 9)$ to $(3, 9)$.
- This component is parametrized by $x = -3 + 6t$, $y = 9$ for $0 \le t \le 1$. On this component we have $f(-3 + 6t, 9) = 36t - 18$, which has no critical point.
- Thus, we only get the endpoints $(-3, 9)$ and $(3, 9)$.

Optimization on a Region, XVI

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

- Component #2, the parabolic arc $(-3, 9)$ to $(3, 9)$.

Optimization on a Region, XVI

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

- Component #2, the parabolic arc $(-3, 9)$ to $(3, 9)$.
- This component is parametrized by $x = t$, $y = t^2$ for $-3 \leq t \leq 3$.
- On this component we have $f(t, t^2) = t^3 - 3t$.
- Then $\frac{df}{dt} = 3t^2 - 3$ which is zero for $t = -1, 1$.
- These yield boundary-critical points $(x, y) = (-1, 1)$ and $(1, 1)$.
- We also have to include the endpoints $(-3, 9)$, $(3, 9)$, though we did this already in the other component.

Optimization on a Region, XVII

Example: Find the absolute maximum and minimum of $f(x, y) = xy - 3x$ on the region with $x^2 \leq y \leq 9$.

- Our full list of points to analyze is $(0, 3)$, $(-3, 9)$, $(3, 9)$, $(-1, 1)$, $(1, 1)$.
- We have $f(0, 3) = 0$, $f(-3, 9) = -18$, $f(3, 9) = 18$, $f(-1, 1) = 2$, and $f(1, 1) = -2$.
- Thus, the minimum value is -18 occurring at $(-3, 9)$ while the maximum value is 18, occurring at $(3, 9)$.

Summary

We discussed some optimization problems.

We discussed how to find the minimum and maximum values of a function on a region.

Next lecture: Review for Midterm 1 (part 1).

This was the last of the material for Midterm 1. Next week's lectures will be devoted to exam review. Also, note that there is no class on Monday the 15th.