# Math 2321 (Multivariable Calculus)

## Lecture #10 of 38 ~ February 10, 2021

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Critical Points, Minima, and Maxima

- Critical Points
- Minima, Maxima, Saddle Points
- Classification of Critical Points

This material represents §2.5.1 from the course notes.

# Motivation

Now that we have developed the basic ideas of derivatives for functions of several variables, we will tackle one of the primary motivating questions for the development of the derivative: finding minima and maxima of functions of several variables.

- We will primarily discuss functions of two variables, because there is a nice criterion for deciding whether a critical point is a minimum or a maximum in that situation.
- Classifying critical points for functions of more than two variables requires some results from linear algebra, so we will not treat functions of more than two variables.
- We will then discuss various different flavors of optimization problems.

# Critical Points, I

We would first like to determine where a function $f$ can have a minimum or maximum value.

- We know that the gradient vector $\nabla f$, evaluated at a point $P$, gives a vector pointing in the direction along which $f$ increases most rapidly at $P$.
- If $\nabla f(P) \neq \mathbf{0}$, then $f$ cannot have a local extreme point at $P$, since $f$ increases along $\nabla f(P)$ and decreases along $-\nabla f(P)$.
- Thus, extreme points can only occur at points where the gradient $\nabla f$ is the zero vector, or if it is undefined.

# Critical Points, II

**Definition**

*A* critical point *of a function* $f(x, y)$ *is a point where* $\nabla f$ *is zero or undefined. Equivalently,* $(x_0, y_0)$ *is a critical point if* $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$*, or either* $f_x(x_0, y_0)$ *or* $f_y(x_0, y_0)$ *is undefined.*

We have a similar definition for functions of 3+ variables.

By the remarks on the previous slide, a local minimum or maximum of a function of several variables can only occur at a critical point. (Note that this is the same state of affairs as with functions of one variable.)

Most critical points will arise from places where all the partial derivatives are zero, since we will rarely encounter functions (e.g., absolute values) that are defined but whose derivatives are not.

# Critical Points, III

Solving systems of equations in several variables (to find the critical points) can sometimes be quite tricky. Some tips:

1. Identify an equation that you can solve for one variable in terms of the other (or others). Use it to eliminate that variable from the other equations, and repeat until you get an equation in one variable.
2. Factor an equation and break into cases. If you have an expression like $[\text{something}] \cdot [\text{another thing}] = 0$ then you know one of the two terms must be zero. This will give you two separate cases to analyze.
3. Manipulate the equations algebraically: add, subtract, multiply, or divide them.

In some cases, one of the equations may be much easier to work with than the other(s). Start with whichever one seems simplest.

# Critical Points, IV

Example: Find all critical points for each given function:

1. $f(x, y) = x^2 + y^2$.
2. $g(x, y) = x^2 + 2x - y^2 - 6y + 4$.
3. $p(x, y) = x^3 + y^3 - 3xy$.
4. $q(x, y) = x^2 + 4xy + 2y^2 + 6x - 4y + 3$.
5. $j(x, y) = x^2y^2 - x^2 - y^2$.
6. $t(x, y) = y\cos(x)$.

- In each case, the partial derivatives are both defined everywhere, so we only need to find where they are both zero.

# Critical Points, V

Example: Find all critical points for each given function:

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# Critical Points, V

Example: Find all critical points for each given function:

1. $f(x, y) = x^2 + y^2$.
   - We have $f_x = 2x$ and $f_y = 2y$.
   - Clearly $f_x = 0$ requires $x = 0$ while $f_y = 0$ requires $y = 0$.
   - Therefore, we get one critical point: $(x, y) = (0, 0)$.
2. $g(x, y) = x^2 + 2x - y^2 - 6y + 4$.

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2. $g(x, y) = x^2 + 2x - y^2 - 6y + 4$.

- We have $g_x = 2x + 2$ and $g_y = -2y - 6$.
- Then $g_x = 0$ requires $x = -1$ while $g_y = 0$ requires $2y = -6$ so $y = -3$.
- Therefore, we get one critical point: $(x, y) = (-1, -3)$.

# Critical Points, VI

Example: Find all critical points for each given function:

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# Critical Points, VI

Example: Find all critical points for each given function:

3. $p(x, y) = x^3 + y^3 - 3xy$.

- We have $p_x = 3x^2 - 3y$ and $p_y = 3y^2 - 3x$.
- So we get the equations $3x^2 - 3y = 0$ and $3y^2 - 3x = 0$.
- Neither equation gives us a value for $x$ or $y$ directly.
- But we can solve the first equation for $y$ in terms of $x$:

# Critical Points, VI

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- So we get the equations $3x^2 - 3y = 0$ and $3y^2 - 3x = 0$.
- Neither equation gives us a value for $x$ or $y$ directly.
- But we can solve the first equation for $y$ in terms of $x$:this gives $y = x^2$.
- Now plugging into the second equation yields $3(x^2)^2 - 3x = 0$, so that $3x^4 - 3x = 0$.
- Factoring gives $3x(x^3 - 1) = 0$, which has the solutions $x = 0$ and $x = 1$.
- If $x = 0$, then $y = x^2 = 0$, so we get the point $(0, 0)$.
- If $x = 1$, then $y = x^2 = 1$, so we get the point $(1, 1)$.
- In total, there are two critical points: $(0, 0)$ and $(1, 1)$.

# Critical Points, VII

Example: Find all critical points for each given function:

4. $q(x, y) = x^2 + 4xy + 2y^2 + 6x - 4y + 3.$

# Critical Points, VII

Example: Find all critical points for each given function:

4. $q(x, y) = x^2 + 4xy + 2y^2 + 6x - 4y + 3$.

- We have $q_x = 2x + 4y + 6$ and $q_y = 4x + 4y - 4$.
- So we get the equations $2x + 4y + 6 = 0$ and $4x + 4y - 4 = 0$.
- Neither equation gives us a value for $x$ or $y$ directly.
- But we can solve the second equation for $y$ in terms of $x$:

# Critical Points, VII

Example: Find all critical points for each given function:

4. $q(x, y) = x^2 + 4xy + 2y^2 + 6x - 4y + 3$.

- We have $q_x = 2x + 4y + 6$ and $q_y = 4x + 4y - 4$.
- So we get the equations $2x + 4y + 6 = 0$ and $4x + 4y - 4 = 0$.
- Neither equation gives us a value for $x$ or $y$ directly.
- But we can solve the second equation for $y$ in terms of $x$:this gives $y = 1 - x$.
- Now plugging into the first equation yields $2x + 4(1 - x) + 6 = 0$, so that $10 - 2x = 0$ and thus $x = 5$.
- Then $y = 1 - x = -4$.
- Therefore, there is one critical point: $(x, y) = (5, -4)$.

There are many other ways to solve this system: for example, we could have solved the first equation for $y$ in terms of $x$, or for $x$ in terms of $y$, or we could have subtracted the two equations.

# Critical Points, VIII

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- We have $j_x = 2xy^2 - 2x$ and $j_y = 2x^2y - 2y$.
- So we get the equations $2xy^2 - 2x = 0$ and $2x^2y - 2y = 0$.
- We can factor the first equation:

# Critical Points, VIII

Example: Find all critical points for each given function:

5. $j(x, y) = x^2y^2 - x^2 - y^2$.
   - We have $j_x = 2xy^2 - 2x$ and $j_y = 2x^2y - 2y$.
   - So we get the equations $2xy^2 - 2x = 0$ and $2x^2y - 2y = 0$.
   - We can factor the first equation: $2x(y^2 - 1) = 0$.
   - This means either $2x = 0$ (so that $x = 0$) or $y^2 - 1 = 0$ (so that $y = 1$ or $y = -1$).
   - If $x = 0$, then the second equation gives $-2y = 0$ so that $y = 0$. We get the point $(0, 0)$.
   - If $y = 1$, then the second equation gives $2x^2 - 2 = 0$ so that $x = 1$ or $x = -1$. We get points $(1, 1)$ and $(-1, 1)$.
   - If $y = -1$, then the second equation gives $-2x^2 + 2 = 0$ so that $x = 1$ or $x = -1$. We get points $(1, -1)$ and $(-1, -1)$.
   - In total we get five critical points: $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$.

# Critical Points, IX

Example: Find all critical points for each given function:

6. $t(x, y) = y \cos(x)$.

# Critical Points, IX

<u>Example</u>: Find all critical points for each given function:

6. $t(x, y) = y\cos(x)$.

- We have $t_x = -y\sin(x)$ and $t_y = \cos(x)$.
- So we get the equations $-y\sin(x) = 0$ and $\cos(x) = 0$.
- The second equation implies $x = \pi/2 + \pi k$ for some integer $k$.
- Then since $\cos(\pi/2 + \pi k) = (-1)^k$, the first equation requires $y = 0$.
- Therefore, the critical points are $(x, y) = (\pi/2 + \pi k, 0)$ for any integer $k$. (Note that there are infinitely many of them!)

# Minima, Maxima, and Saddles, I

We have various different types of critical points:

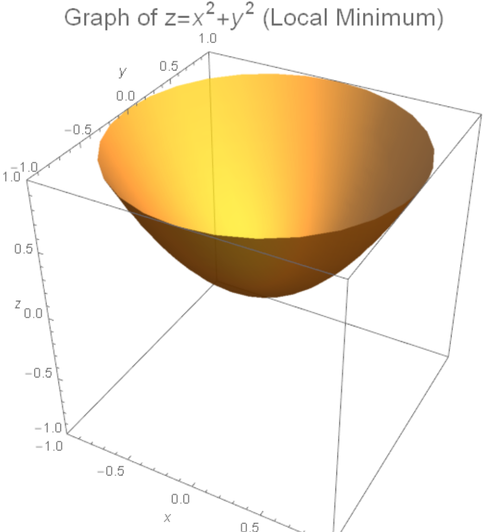
**Definition**

*A local minimum is a point where $f$ is nearby always bigger.*
*A local maximum is a point where $f$ is nearby always smaller.*
*A saddle point is a critical point where $f$ nearby is bigger in some directions and smaller in others.*

From our earlier discussion, local minima and local maxima always occur at critical points.

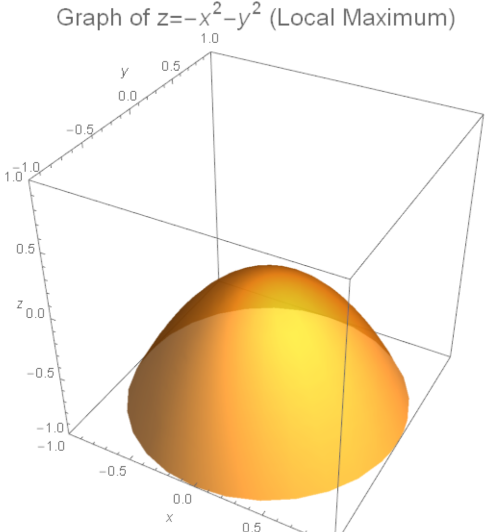
# Minima, Maxima, and Saddles, II

Example: $f(x, y) = x^2 + y^2$ has a local minimum at $(0, 0)$:



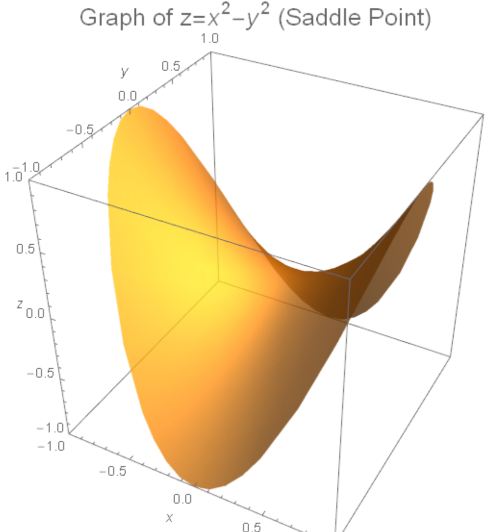
# Minima, Maxima, and Saddles, III

Example: $f(x,y) = -x^2 - y^2$ has a local maximum at $(0,0)$:



# Minima, Maxima, and Saddles, IV

Example: $f(x, y) = x^2 - y^2$ has a saddle point at $(0, 0)$:



# Minima, Maxima, and Saddles, V

Local minima and local maxima are (presumably) familiar from the one-variable setting. Saddle points, however, are a new kind of critical point.

- A saddle point will look like a local minimum along some directions and a local maximum along other directions.
- For example, $f(x, y) = x^2 - y^2$ looks like a minimum in the $x$-direction (as $x$ varies and $y$ is held fixed at 0, the function is $f(x, 0) = x^2$) but a maximum in the $y$-direction (as $y$ varies and $x$ is held fixed at 0, the function is $f(0, y) = -y^2$).
- In contrast, a local minimum looks like a minimum in every direction, while a local maximum looks like a maximum in every direction.

# Classification of Critical Points, I

Once we can identify a function's critical points, we would like to know whether those points actually are minima or maxima of $f$. We can determine this using a quantity called the discriminant:

**Definition**

*The discriminant (also called the Hessian) at a critical point is the value $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$, where each of the second-order partials is evaluated at the critical point.*

One way to remember the definition of the discriminant is as the determinant of the matrix of the four second-order partials: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. (We are implicitly using the fact that $f_{xy} = f_{yx}$.)

# Classification of Critical Points, II

Example: Find $D$ at the critical points $(0,0)$ and $(1,1)$ for $f(x,y) = x^3 + y^3 - 3xy$.

Remember that $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.

# Classification of Critical Points, II

Example: Find $D$ at the critical points $(0,0)$ and $(1,1)$ for $f(x,y) = x^3 + y^3 - 3xy$.

Remember that $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$.

- We have $f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$, so $f_{xx} = 6x$, $f_{xy} = -3$, and $f_{yy} = 6y$.
- Therefore, $D = (6x)(6y) - (-3)^2$.
- Thus, at $(x,y) = (0,0)$ we get $D = (0)(0) - 9 = -9$.
- Also, at $(x,y) = (1,1)$ we get $D = (6)(6) - 9 = 27$.

# Classification of Critical Points, II

Our main result is that we can use $D$ to classify critical points:

**Theorem (Second Derivatives Test)**

*Suppose $P$ is a critical point of $f(x, y)$, and let $D$ be the value of the discriminant $f_{xx} f_{yy} - f_{xy}^2$ at $P$.*
*If $D > 0$ and $f_{xx} > 0$, then the critical point is a local minimum.*
*If $D > 0$ and $f_{xx} < 0$, then the critical point is a local maximum.*
*If $D < 0$, then the critical point is a saddle point.*
*If $D = 0$, then the test is inconclusive.*

Note that if $D > 0$ then $f_{xx}$ cannot be zero, because $D = -f_{xy}^2$ in that case.

# Classification of Critical Points, III

Proof (outline):

- Assume for simplicity that $P$ is at the origin.
- Then by our results on Taylor series, the function $f(x,y) - f(P)$ will be closely approximated by the polynomial $ax^2 + bxy + cy^2$, where $a = \frac{1}{2} f_{xx}$, $b = f_{xy}$, and $c = \frac{1}{2} f_{yy}$.
- If $D \neq 0$, then the behavior of $f(x,y)$ near the critical point $P$ will be the same as that quadratic polynomial.
- Completing the square and examining whether the resulting quadratic polynomial has any real roots and whether it opens or downwards yields the test.
- The behavior of the roots of the quadratic polynomial $ax^2 + bxy + cy^2$ is determined by its discriminant $b^2 - 4ac$.
- Here, $b^2 - 4ac = f_{xy}^2 - f_{xx} f_{yy} = -D$. (That's why $D$ is called the discriminant. Though yes, there is the minus sign....)

# Classification of Critical Points, IV

Example: Classify the type of critical point that $f(x, y) = x^2 + y^2$ has at the origin $(0, 0)$.

# Classification of Critical Points, IV

Example: Classify the type of critical point that $f(x, y) = x^2 + y^2$ has at the origin $(0, 0)$.

- We saw earlier that $(0, 0)$ is a critical point of this function.
- To classify it, we compute $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = 2$.
- Then $D = f_{xx}f_{yy} - (f_{xy})^2 = 2 \cdot 2 - 0^2 = 4$. (Here, $D$ is constant, but normally we would need to evaluate it at our point.)
- So, by the second derivatives test, since $D > 0$ and $f_{xx} > 0$ at $(0, 0)$, we see that $(0, 0)$ is a local minimum.

# Classification of Critical Points, V

Example: Classify the two critical points $(0, 0)$ and $(1, 1)$ for $p(x, y) = x^3 + y^3 - 3xy$.

- We saw earlier that $(0, 0)$ and $(1, 1)$ are the critical points of this function and that $D = (6x)(6y) - (-3)^2$.

# Classification of Critical Points, V

Example: Classify the two critical points $(0, 0)$ and $(1, 1)$ for $p(x, y) = x^3 + y^3 - 3xy$.

- We saw earlier that $(0, 0)$ and $(1, 1)$ are the critical points of this function and that $D = (6x)(6y) - (-3)^2$.
- At $(0, 0)$, we have $D = -9$, so $(0, 0)$ is a saddle point.
- At $(1, 1)$, we have $D = (6)(6) - (-3)^2 = 27$, and also $f_{xx} = 6x = 6 > 0$, so $(1, 1)$ is a local minimum.

# Classification of Critical Points, VI

Example: Classify the critical point $(5, -4)$ for $q(x, y) = x^2 + 4xy + 2y^2 + 6x - 4y + 3$.

# Classification of Critical Points, VI

Example: Classify the critical point $(5, -4)$ for $q(x, y) = x^2 + 4xy + 2y^2 + 6x - 4y + 3$.

- We have $q_x = 2x + 4y + 6$ and $q_y = 4x + 4y - 4$ so $q_{xx} = 2$, $q_{xy} = 4$, and $q_{yy} = 4$.
- Therefore, $D = (2)(4) - 4^2 = -8$. Here, $D$ is constant.
- Then at the critical point, we have $D = -8$, so by the second derivatives test, $(5, -4)$ is a saddle point.

# Classification of Critical Points, VII

Example: For $f(x, y) = 3x^2 + 2y^3 - 6xy$, find the critical points of $f$ and classify them as minima, maxima, or saddle points.

## Classification of Critical Points, VII

Example: For $f(x, y) = 3x^2 + 2y^3 - 6xy$, find the critical points of $f$ and classify them as minima, maxima, or saddle points.

- First, $f_x = 6x - 6y$ and $f_y = 6y^2 - 6x$. They are both defined everywhere so we need only find where they are both zero.
- Next, we can see that $f_x$ is zero only when $y = x$.
- Then the equation $f_y = 0$ becomes $6x^2 - 6x = 0$, which by factoring we can see has solutions $x = 0$ or $x = 1$.
- Since $y = x$, we see $(0, 0)$ and $(1, 1)$ are the critical points.

# Classification of Critical Points, VII

Example: For $f(x,y) = 3x^2 + 2y^3 - 6xy$, find the critical points of $f$ and classify them as minima, maxima, or saddle points.

- First, $f_x = 6x - 6y$ and $f_y = 6y^2 - 6x$. They are both defined everywhere so we need only find where they are both zero.
- Next, we can see that $f_x$ is zero only when $y = x$.
- Then the equation $f_y = 0$ becomes $6x^2 - 6x = 0$, which by factoring we can see has solutions $x = 0$ or $x = 1$.
- Since $y = x$, we see $(0,0)$ and $(1,1)$ are the critical points.
- To classify them, we compute $f_{xx} = 6$, $f_{xy} = -6$, and $f_{yy} = 12y$. Then $D(0,0) = 6 \cdot 0 - (-6)^2 < 0$ and $D(1,1) = 6 \cdot 12 - (-6)^2 > 0$. Also, $f_{xx} > 0$ at $(1,1)$.
- So, by the second derivatives test, $(0,0)$ is a saddle point and $(1,1)$ is a local minimum.

# Classification of Critical Points, VIII

Example: Let $g(x, y) = x^3y - 3xy^3 + 8y$.

1. Find the critical points of $g$.
2. Classify each critical point as a local minimum, local maximum, or saddle point.

# Classification of Critical Points, VIII

Example: Let $g(x, y) = x^3y - 3xy^3 + 8y$.

1. Find the critical points of $g$.
2. Classify each critical point as a local minimum, local maximum, or saddle point.

- Since $g_x$ and $g_y$ are defined everywhere, the critical points are obtained by solving $g_x = g_y = 0$.
- Then we use the second derivatives test to classify the types.

# Classification of Critical Points, IX

Example: Let $g(x, y) = x^3y - 3xy^3 + 8y$.

1. Find the critical points of $g$.

# Classification of Critical Points, IX

Example: Let $g(x, y) = x^3y - 3xy^3 + 8y$.

1. Find the critical points of $g$.

- We have $g_x = 3x^2y - 3y^3$ and $g_y = x^3 - 9xy^2 + 8$.
- Since $g_x = 3y(x^2 - y^2) = 3y(x + y)(x - y)$, we see that $g_x = 0$ precisely when $y = 0$ or $y = x$ or $y = -x$.
- If $y = 0$, then $g_y = 0$ implies $x^3 + 8 = 0$, so that $x = -2$. This yields the point $(x, y) = (-2, 0)$.
- If $y = x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, 1)$.
- If $y = -x$, then $g_y = 0$ implies $-8x^3 + 8 = 0$, so that $x = 1$. This yields the point $(x, y) = (1, -1)$.
- Thus, $(-2, 0)$, $(1, 1)$, and $(1, -1)$ are the critical points.

# Classification of Critical Points, VIII

Example: Let $g(x, y) = x^3y - 3xy^3 + 8y$.

2. Classify each critical point as a local minimum, local maximum, or saddle point.

- $(-2, 0)$, $(1, 1)$, and $(1, -1)$ are the critical points.

# Classification of Critical Points, VIII

Example: Let $g(x, y) = x^3y - 3xy^3 + 8y$.

2. Classify each critical point as a local minimum, local maximum, or saddle point.

- $(-2, 0)$, $(1, 1)$, and $(1, -1)$ are the critical points.
- To classify them, we compute $g_{xx} = 6xy$, $g_{xy} = 3x^2 - 9y^2$, and $g_{yy} = -18xy$.
- Therefore, $D = (6xy)(-18xy) - (3x^2 - 9y^2)^2$.
- Then $D(-2, 0) = 0 \cdot 0 - (12)^2 < 0$,
  $D(1, 1) = 6 \cdot (-18) - (-6)^2 < 0$, and
  $D(1, -1) = (-6) \cdot (18) - (-6)^2 < 0$.
- So, by the second derivatives test, $(-2, 0)$, $(1, 1)$, and $(1, -1)$ are all saddle points.

# Summary

We introduced critical points and discussed how to find them.

We discussed how to classify critical points as local minima, local maxima, or saddle points.

Next lecture: Applied optimization, optimization of a function on a region.