Math 2321 (Multivariable Calculus)

Lecture #9 of 38 ~ February 8th, 2021

The Chain Rule + Implicit Differentiation

- The Chain Rule
- Implicit Differentiation

This material represents §2.3.1-2.3.2 from the course notes.

Exam Information, I

Midterm 1 is next Friday, February 19th.

- The exam has the same format as if it were being given in class, in person.
- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- There are approximately 6 pages of material, about 1/5 multiple choice and the rest free response.
- I have set up a selection of various time windows for you to take the exam. You will select one of them ahead of time and then take the exam at that time.

Next week's lectures (Wed Feb 17th, Thu Feb 18th) will be devoted to review. I will go over problems from the sheet of review problems posted on the course webpage.

Exam Information, II

I have set up a Piazza poll for you to select your exam window.

- The "official" exam time limit is 65 minutes. I give you 25 extra minutes of working time, so you really have 90 minutes. To this are added 30 minutes of turnaround time (for downloading, printing, scanning, and uploading).
- Thus, the exam windows are 120 minutes in length (180 if you have a DRC-approved testing-time accommodation).
- The Canvas assignment will disappear after your exam window finishes. **After that time, you cannot submit via Canvas.**
- The windows are as follows: (Friday) 10:30am-12:30pm, 1:30pm-3:30pm, 4:30pm-6:30pm, 8pm-10pm (Saturday) 1am-3am, 4am-6am, 9am-11am.

Please make your selection by **Wednesday February 17th**. I will send confirmations via Canvas notification that evening.

Exam Information, III

Some notes on the exam format:

- You are allowed to use calculators / equivalent computing technology on the exam.
- You are also allowed to use notes. Normally this would not be allowed, however it does not seem reasonable to disallow notes when you are taking the exam remotely.
- However, you must show all relevant details and identify whenever you used a calculator to do a calculation. You are expected to justify all calculations and show all work. Correct answers without appropriate work may not receive full credit.
- Collaboration of other kind is not allowed – all work must be your own.

Exam Information, IV

I will post the full topics list for the exam when we get into the review material next week, but the exam covers chapter 1 (Vectors and 3D Geometry) and all but the very end of chapter 2 (Partial Derivatives), up through §2.5.2, representing WeBWorKs 1-4 and course lectures 1-11.

Any questions about exam logistics?

The Chain Rule, I

In many situations, we have functions that depend on variables indirectly, and we often need to determine the precise nature of the dependence of one variable on another.

- To do this, we want to generalize the chain rule for functions of several variables.
- With functions of several variables, each of which is defined in terms of other variables (for example, $f(x, y)$ where x and y are themselves functions of s and t), we will recover a version of the chain rule specific to the relations between the variables involved.

The Chain Rule, II

Recall that the chain rule for functions of one variable states that $\frac{dg}{dx} = \frac{dg}{dy} \cdot \frac{dy}{dx}$, if g is a function of y and y is a function of x.

- Roughly speaking, you can interpret the one-variable chain rule as follows: g depends y which in turn depends on x
- Therefore, if we change x, this will cause y to change, which will in turn cause g to change.
- If we change x by Δx, y will change by roughly $\Delta y \approx \frac{dy}{dx} \Delta x$.
- If we change y by Δy, g will change by roughly $\Delta g \approx \frac{dg}{dy} \Delta y$.
- So now we just plug these expressions into one another to see that $\Delta g \approx \frac{dg}{dy} \Delta y \approx \frac{dg}{dy} \frac{dy}{dx} \Delta x$, so that $\frac{\Delta g}{\Delta x} \approx \frac{dg}{dy} \frac{dy}{dx}$.
- Then, taking the limit as $\Delta x \to 0$, we end up with the desired equality.

The Chain Rule, III

The various chain rules for functions of more than one variable have a similar form, but they will involve more terms that depend on the relationships between the variables.

- However, they all have a similar sort of form and interpretation as the one-variable chain rule.
- What we do is trace how much each of the intermediate variables will change as a result of changing our variable of interest, and then add up all of the total contributions.

The Chain Rule, III

Here is the outline for finding $\frac{df}{dt}$ for a function $f(x, y)$ where x and y are both functions of t:

- If t changes to $t + \Delta t$, then x changes to $x + \Delta x$ and y changes to $y + \Delta y$.
- Then $\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y)$ is roughly equal to the directional derivative of $f(x, y)$ in the direction of the (non-unit) vector $\langle \Delta x,\ \Delta y \rangle$.
- We know that this directional derivative is equal to the dot product of $\langle \Delta x,\ \Delta y \rangle$ with the gradient $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$.
- Then $\frac{\Delta f}{\Delta t} \approx \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right\rangle$.
- Taking the limit as $\Delta t \to 0$ yields $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$.

The Chain Rule, IV

We obtain similar sorts of statements for other arrangements of variable dependencies.

- Rather than try to describe these one at a time, we will give a general procedure for generating the statement of the chain rule specific to any particular set of dependencies of variables.
- As an unrelated remark, in the situation where a function f depends on only one independent variable t, we will write df/dt rather than $\partial f/\partial t$ because f ultimately depends on only the single variable t, so we are actually computing a single-variable derivative and not a partial derivative.
  (In fact, this notation was already used on the previous slide.)

The Chain Rule, V

The method is to draw a "tree diagram" as follows:

1. Start with the initial function f, and draw an arrow pointing from f to each of the variables it depends on.
2. For each variable listed, draw new arrows branching from that variable to any other variables they depend on. Repeat the process until all dependencies are shown in the diagram.
3. Associate each arrow from one variable to another with the derivative $\frac{\partial[\text{top}]}{\partial[\text{bottom}]}$.
4. To write the version of the chain rule that gives the derivative $\partial v_1/\partial v_2$ for any variables v_1 and v_2 in the diagram (where v_2 depends on v_1), first find all paths from v_1 to v_2.
5. For each path from v_1 to v_2, multiply all of the derivatives that appear in each path from v_1 to v_2. Finally, sum the results over all of the paths: this is $\partial v_1/\partial v_2$.

The Chain Rule, VI

Example: State the chain rule that computes $\frac{df}{dt}$ for the function $f(x, y, z)$, where each of x, y, and z is a function of the variable t.

The Chain Rule, VI

<u>Example</u>: State the chain rule that computes $\frac{df}{dt}$ for the function $f(x, y, z)$, where each of x, y, and z is a function of the variable t.

- First, we draw the tree diagram: $\begin{array}{ccccc} & & f & & \\ & \swarrow & \downarrow & \searrow & \\ x & & y & & z \\ \downarrow & & \downarrow & & \downarrow \\ t & & t & & t \end{array}$.
- In the tree diagram, there are 3 paths from f to t: they are $f \to x \to t$, $f \to y \to t$, and $f \to z \to t$.
- The path $f \to x \to t$ gives $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t}$, while the path $f \to y \to t$ gives $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$, and the path $f \to z \to t$ gives $\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$.
- So, the statement is $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$.

The Chain Rule, VII

Example: State the chain rule that computes $\frac{df}{dt}$ for the function $f(x, y, z)$, where each of x, y, and z is a function of the variable t.

- The chain rule says $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt} + \dfrac{\partial f}{\partial z} \cdot \dfrac{dz}{dt}$.
- You can interpret this statement as saying that the total change in f is the sum of three components:
  1. The change $\dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt}$ in f resulting from the change in x.
  2. The change $\dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt}$ in f resulting from the change in y.
  3. The change $\dfrac{\partial f}{\partial z} \cdot \dfrac{dz}{dt}$ in f resulting from the change in z.

The Chain Rule, VII

Example: State the chain rule that computes $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ for the function $f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$ are both functions of s and t.

The Chain Rule, VII

<u>Example</u>: State the chain rule that computes $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ for the function $f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$ are both functions of s and t.

- First, we draw the tree diagram:

$$\begin{array}{ccccccccc} & & & & f & & & & \\ & & & \swarrow & & \searrow & & & \\ & & x & & & & y & & \\ & \swarrow & \downarrow & & & & \downarrow & \searrow & \\ s & & t & & & & s & & t \end{array}.$$

- In this diagram, there are 2 paths from f to s: they are $f \to x \to s$ and $f \to y \to s$, and also two paths from f to t: $f \to x \to t$ and $f \to y \to t$.

The Chain Rule, VIII

Example: State the chain rule that computes $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ for the function $f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$ are both functions of s and t.

- The path $f \to x \to t$ gives the product $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t}$, while the path $f \to y \to t$ gives the product $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$.
- Similarly, the path $f \to x \to s$ gives the product $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s}$, while $f \to y \to s$ gives the product $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$.

The Chain Rule, VIII

Example: State the chain rule that computes $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$ for the function $f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$ are both functions of s and t.

- The path $f \to x \to t$ gives the product $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t}$, while the path $f \to y \to t$ gives the product $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$.
- Similarly, the path $f \to x \to s$ gives the product $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s}$, while $f \to y \to s$ gives the product $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$.
- Thus, the two statements of the chain rule here are $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$ and $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$.

The Chain Rule, IX

Example: For $f(x, y) = x^2 + y^2$, with $x = t^2$ and $y = t^4$, find $\frac{df}{dt}$, both directly and via the chain rule.

The Chain Rule, IX

<u>Example</u>: For $f(x,y) = x^2 + y^2$, with $x = t^2$ and $y = t^4$, find $\frac{df}{dt}$, both directly and via the chain rule.

- In this instance, the multivariable chain rule says that $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$.
- Computing the derivatives shows $\frac{df}{dt} = (2x) \cdot (2t) + (2y) \cdot (4t^3)$.
- Plugging in $x = t^2$ and $y = t^4$ yields $\frac{df}{dt} = (2t^2) \cdot (2t) + (2t^4) \cdot (4t^3) = 4t^3 + 8t^7$.
- To do this directly, we would plug in $x = t^2$ and $y = t^4$: this gives $f(x,y) = t^4 + t^8$, so that $\frac{df}{dt} = 4t^3 + 8t^7$.
- Of course, we obtain the same answer either way!

The Chain Rule, X

Example: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

1. Find $\dfrac{\partial f}{\partial s}$.
2. Find $\dfrac{\partial f}{\partial t}$.

The Chain Rule, X

Example: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

1. Find $\dfrac{\partial f}{\partial s}$.
2. Find $\dfrac{\partial f}{\partial t}$.

- We just need to write down and then apply the appropriate version of the chain rule.

The Chain Rule, XI

<u>Example</u>: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

1. Find $\dfrac{\partial f}{\partial s}$.

The Chain Rule, XI

<u>Example</u>: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

1. Find $\dfrac{\partial f}{\partial s}$.

- By the chain rule we have $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial s} = (2x) \cdot (2s) + (2y) \cdot (3s^2)$.
- Plugging in $x = s^2 + t^2$ and $y = s^3 + t^4$ yields

$$\begin{aligned}\frac{\partial f}{\partial s} &= (2s^2 + 2t^2) \cdot (2s) + (2s^3 + 2t^4) \cdot (3s^2) \\ &= 4s^3 + 4st^2 + 6s^5 + 6s^2 t^4.\end{aligned}$$

The Chain Rule, XII

Example: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

2. Find $\dfrac{\partial f}{\partial t}$.

The Chain Rule, XII

Example: Let $f(x, y) = x^2 + y^2$, where $x = s^2 + t^2$ and $y = s^3 + t^4$.

2. Find $\dfrac{\partial f}{\partial t}$.

- By the chain rule we have $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial t} = (2x) \cdot (2t) + (2y) \cdot (4t^3)$.
- Plugging in $x = s^2 + t^2$ and $y = s^3 + t^4$ yields

$$\begin{aligned} \frac{\partial f}{\partial s} &= (2s^2 + 2t^2) \cdot (2t) + (2s^3 + 2t^4) \cdot (4t^3) \\ &= 4s^2 t + 4t^3 + 8s^3 t^3 + 8t^7. \end{aligned}$$

The Chain Rule, Lucky XIII

<u>Example</u>: Suppose x and y are functions of s and t with $x(1,5) = 2$, $y(1,5) = -2$, and that f has partial derivatives below.

$\frac{\partial f}{\partial x}(1,5) = 7$	$\frac{\partial f}{\partial y}(1,5) = -6$	$\frac{\partial f}{\partial x}(2,-2) = 1$	$\frac{\partial f}{\partial y}(2,-2) = -4$
$\frac{\partial x}{\partial s}(1,5) = 3$	$\frac{\partial x}{\partial t}(1,5) = 2$	$\frac{\partial y}{\partial s}(1,5) = 4$	$\frac{\partial y}{\partial t}(1,5) = -2$

1. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (1,5)$.
2. Find $\frac{\partial f}{\partial t}$ at $(s,t) = (1,5)$.

The Chain Rule, Lucky XIII

Example: Suppose x and y are functions of s and t with $x(1,5) = 2$, $y(1,5) = -2$, and that f has partial derivatives below.

$\frac{\partial f}{\partial x}(1,5) = 7$	$\frac{\partial f}{\partial y}(1,5) = -6$	$\frac{\partial f}{\partial x}(2,-2) = 1$	$\frac{\partial f}{\partial y}(2,-2) = -4$
$\frac{\partial x}{\partial s}(1,5) = 3$	$\frac{\partial x}{\partial t}(1,5) = 2$	$\frac{\partial y}{\partial s}(1,5) = 4$	$\frac{\partial y}{\partial t}(1,5) = -2$

1. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (1,5)$.
2. Find $\frac{\partial f}{\partial t}$ at $(s,t) = (1,5)$.

- This problem is an application of the chain rule. We need only write down the appropriate chain rule and then plug in the proper values (namely, $s = 1$ and $t = 5$).
- However, the notation for this problem is tricky. The variables for f are x and y, and when $s = 1$ and $t = 5$, we have $x = 2$ and $y = -2$. So we want to use the two entries on the top right, *not* the top left.

The Chain Rule, XIV

Example: Suppose x and y are functions of s and t with $x(1,5) = 2$, $y(1,5) = -2$, and that f has partial derivatives below.

$\frac{\partial f}{\partial x}(1,5) = 7$	$\frac{\partial f}{\partial y}(1,5) = -6$	$\frac{\partial f}{\partial x}(2,-2) = 1$	$\frac{\partial f}{\partial y}(2,-2) = -4$
$\frac{\partial x}{\partial s}(1,5) = 3$	$\frac{\partial x}{\partial t}(1,5) = 2$	$\frac{\partial y}{\partial s}(1,5) = 4$	$\frac{\partial y}{\partial t}(1,5) = -2$

1. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (1,5)$.

The Chain Rule, XIV

Example: Suppose x and y are functions of s and t with $x(1,5) = 2$, $y(1,5) = -2$, and that f has partial derivatives below.

$\frac{\partial f}{\partial x}(1,5) = 7$	$\frac{\partial f}{\partial y}(1,5) = -6$	$\frac{\partial f}{\partial x}(2,-2) = 1$	$\frac{\partial f}{\partial y}(2,-2) = -4$
$\frac{\partial x}{\partial s}(1,5) = 3$	$\frac{\partial x}{\partial t}(1,5) = 2$	$\frac{\partial y}{\partial s}(1,5) = 4$	$\frac{\partial y}{\partial t}(1,5) = -2$

1. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (1,5)$.

- By the chain rule, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$.
- Setting $(s,t) = (1,5)$ and noting that $x(1,5) = 2$, $y(1,5) = -2$ yields $\frac{\partial f}{\partial s}(1,5) = 1 \cdot 3 + (-4) \cdot 4 = -13$.

The Chain Rule, XV

<u>Example</u>: Suppose x and y are functions of s and t with $x(1,5) = 2$, $y(1,5) = -2$, and that f has partial derivatives below.

$\frac{\partial f}{\partial x}(1,5) = 7$	$\frac{\partial f}{\partial y}(1,5) = -6$	$\frac{\partial f}{\partial x}(2,-2) = 1$	$\frac{\partial f}{\partial y}(2,-2) = -4$
$\frac{\partial x}{\partial s}(1,5) = 3$	$\frac{\partial x}{\partial t}(1,5) = 2$	$\frac{\partial y}{\partial s}(1,5) = 4$	$\frac{\partial y}{\partial t}(1,5) = -2$

2. Find $\frac{\partial f}{\partial t}$ at $(s,t) = (1,5)$.

The Chain Rule, XV

<u>Example</u>: Suppose x and y are functions of s and t with $x(1,5) = 2$, $y(1,5) = -2$, and that f has partial derivatives below.

$\frac{\partial f}{\partial x}(1,5) = 7$	$\frac{\partial f}{\partial y}(1,5) = -6$	$\frac{\partial f}{\partial x}(2,-2) = 1$	$\frac{\partial f}{\partial y}(2,-2) = -4$
$\frac{\partial x}{\partial s}(1,5) = 3$	$\frac{\partial x}{\partial t}(1,5) = 2$	$\frac{\partial y}{\partial s}(1,5) = 4$	$\frac{\partial y}{\partial t}(1,5) = -2$

2. Find $\frac{\partial f}{\partial t}$ at $(s,t) = (1,5)$.

- By the chain rule, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$.
- Setting $(s,t) = (1,5)$ and noting that $x(1,5) = 2$, $y(1,5) = -2$ yields $\frac{\partial f}{\partial t}(1,5) = 1 \cdot 2 + (-4) \cdot (-2) = 10$.

The Chain Rule, XVI

Example: Suppose $f(x, y, z)$ is a function of x, y, z, where z is a function of x and t, and x and y are also both functions of t.

1. State the chain rule for finding $\dfrac{df}{dt}$.

The Chain Rule, XVI

Example: Suppose $f(x, y, z)$ is a function of x, y, z, where z is a function of x and t, and x and y are also both functions of t.

1. State the chain rule for finding $\frac{df}{dt}$.
   - We just need to write down and then apply the appropriate version of the chain rule.
   - Drawing the tree diagram yields paths $f \to x \to t$, $f \to y \to t$, $f \to z \to x \to t$, and $f \to z \to t$.
   - Thus, the statement is $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$.
2. Try it for $f(x, y, z) = x^2 + y^3 + z^4$, $x = t^2$, $y = t^3$, $z = x^2 t$.

The Chain Rule, XVI

Example: Suppose $f(x, y, z)$ is a function of x, y, z, where z is a function of x and t, and x and y are also both functions of t.

1. State the chain rule for finding $\frac{df}{dt}$.
   - We just need to write down and then apply the appropriate version of the chain rule.
   - Drawing the tree diagram yields paths $f \to x \to t$, $f \to y \to t$, $f \to z \to x \to t$, and $f \to z \to t$.
   - Thus, the statement is $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$.
2. Try it for $f(x, y, z) = x^2 + y^3 + z^4$, $x = t^2$, $y = t^3$, $z = x^2 t$.

- $df/dt = (2x)(2t) + (3y^2)(3t^2) + (4z^3)(2xt)(2t) + (4z^3)(x^2)$.

The Chain Rule, XVII

Amusingly, we can actually derive the other differentiation rules using the (multivariable) chain rule!

- To obtain the product rule, let $P(f,g) = fg$, where f and g are both functions of x.
- Then the chain rule says
$$\frac{dP}{dx} = \frac{\partial P}{\partial f} \cdot \frac{df}{dx} + \frac{\partial P}{\partial g} \cdot \frac{dg}{dx} = g \cdot \frac{df}{dx} + f \cdot \frac{dg}{dx}.$$
- If we rewrite this expression using single-variable notation, it reads as the more familiar $(fg)' = f'g + fg'$, which is, indeed, the product rule.
- Likewise, if we set $Q(f,g) = f/g$ where f and g are both functions of x, then applying the chain rule gives
$$\frac{dQ}{dx} = \frac{\partial Q}{\partial f} \cdot \frac{df}{dx} + \frac{\partial Q}{\partial g} \cdot \frac{dg}{dx} = \frac{1}{g} \cdot \frac{df}{dx} - \frac{f}{g^2} \cdot \frac{dg}{dx} = \frac{f'g - fg'}{g^2};$$
this is the quotient rule.

Implicit Differentiation, I

Using the chain rule for several variables, we can give an alternative way to solve problems involving implicit differentiation from calculus of a single variable.

- A typical example is as follows: if y is defined implicitly by $x^3y^2 + \sin(2xy) = 7$, find the derivative $y' = dy/dx$.
- To solve this problem one differentiates both sides with respect to x, using the chain rule to differentiate any terms involving y, and then solves for y'.
- This yields $3x^2y^2 + x^3(2yy') + (2y + 2xy')\cos(2xy) = 0$, so that $y' = -\dfrac{3x^2y^2 + 2y\cos(2xy)}{2x^3y + 2x\cos(2xy)}$.
- Although there is nothing conceptually difficult about this problem, it is often very easy to make algebra mistakes.
- However, implicit differentiation is actually very simple and straightforward if we use partial derivatives.

Implicit Differentiation, II

Here is how to do implicit differentiation with partial derivatives:

Theorem (Implicit Differentiation)

Given an implicit relation $f(x, y) = c$, the implicit single-variable derivative $\frac{dy}{dx} = y'(x)$ is given by $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}$.

The key idea behind this formula is to invoke the multivariable chain rule on the implicit relation $f(x, y) = c$, where we think of y as being a function of x.

Implicit Differentiation, III

Proof:

- Apply the chain rule to the function $f(x, y) = c$, where y is also a function of x.
- The tree diagram is $\begin{array}{ccc} & & f \\ & \swarrow & \downarrow \\ x & & y \\ & & \downarrow \\ & & x \end{array}$, so there are two paths from f to x: $f \to x$ giving $\dfrac{\partial f}{\partial x}$, and $f \to y \to x$ giving $\dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx}$.
- Thus, the chain rule says $\dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx}$.
- However, because $f(x, y) = c$ is a constant function, we have $\dfrac{df}{dx} = 0$. Thus, $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx} = 0$, and so $\dfrac{dy}{dx} = -\dfrac{\partial f / \partial x}{\partial f / \partial y}$.

Implicit Differentiation, IV

Example: If y is defined implicitly by $x^3y^2 + \sin(2xy) = 7$, find the derivative $y' = dy/dx$.

Implicit Differentiation, IV

Example: If y is defined implicitly by $x^3y^2 + \sin(2xy) = 7$, find the derivative $y' = dy/dx$.

- This is the relation $f(x, y) = 7$ for $f(x, y) = x^3y^2 + \sin(2xy)$.
- We have $f_x = 3x^2y^2 + 2y\cos(2xy)$ and $f_y = 2x^3y + 2x\cos(2xy)$.
- So, the formula gives $y' = \dfrac{dy}{dx} = -\dfrac{3x^2y^2 + 2y\cos(2xy)}{2x^3y + 2x\cos(2xy)}$.
- Note that this agrees with the implicit differentiation calculation I did a few slides ago. (You can decide for yourself which one seems easier!)

Implicit Differentiation, V

Example: Find y', if $y^2x^3 + y\,e^x = 2$.

Implicit Differentiation, V

<u>Example</u>: Find y', if $y^2x^3 + y\,e^x = 2$.

- This is the relation $f(x, y) = 2$, where $f(x, y) = y^2x^3 + y\,e^x$.
- We have $f_x = 3y^2x^2 + y\,e^x$, and $f_y = 2yx^3 + e^x$, so the formula gives $y' = \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3y^2x^2 + y\,e^x}{2yx^3 + e^x}$.
- If you want to check this by doing the implicit differentiation directly, here is the calculation:
  - We have $\frac{d}{dx}\left[y^2x^3 + y\,e^x\right] = 0$.
  - Differentiating the left-hand side yields $\left(2yy'x^3 + 3y^2x^2\right) + \left(y'\,e^x + y\,e^x\right) = 0$.
  - Rearranging yields $\left(2yx^3 + e^x\right)y' + \left(3y^2x^2 + y\,e^x\right) = 0$.
  - So $y' = -\frac{3y^2x^2 + y\,e^x}{2yx^3 + e^x}$.

Implicit Differentiation, VI

We can also perform implicit differentiation in the event that there are more than 2 variables.

- For an implicit relation $f(x, y, z) = c$, we can compute implicit derivatives for any of the 3 variables with respect to either of the others, using a chain rule calculation like in the theorem earlier.

Implicit Differentiation, VII

The idea is simply to view any variables other than our target variables as constants.

- So, if we have an implicit relation $f(x,y,z) = c$ and want to compute $\frac{\partial z}{\partial x}$, we view the other variable (in this case y) as constant. Then the chain rule says $\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{f_x}{f_z}$.
- If we wanted to compute $\frac{\partial z}{\partial y}$ then we would view x as a constant, and then $\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{f_y}{f_z}$.
- Similarly, $\frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$, $\frac{\partial y}{\partial z} = -\frac{f_z}{f_y}$, $\frac{\partial x}{\partial y} = -\frac{f_y}{f_x}$, and $\frac{\partial x}{\partial z} = -\frac{f_z}{f_x}$.
- The general form is $\frac{\partial v}{\partial w} = -\frac{f_w}{f_v}$ for any variables v, w.

Implicit Differentiation, VIII

Example: Suppose that x, y, z satisfy $x^2yz + e^x \cos(y) = 3$.

1. If z is defined implicitly in terms of x and y, find $\partial z/\partial x$.

Implicit Differentiation, VIII

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1. If z is defined implicitly in terms of x and y, find $\partial z/\partial x$.
    - The implicit relation is $f(x, y, z) = 3$ where $f(x, y, z) = x^2yz + e^x\cos(y)$. We then compute $f_x = 2xyz + e^x\cos(y)$, $f_y = x^2z - e^x\sin(y)$, $f_z = x^2y$.
    - By the implicit differentiation formulas, $\dfrac{\partial z}{\partial x} = -\dfrac{\partial f/\partial x}{\partial f/\partial z} = -\dfrac{f_x}{f_z} = -\dfrac{2xyz + e^x\cos(y)}{x^2y}$.
2. If y is defined implicitly in terms of x and z, find $\partial y/\partial z$.

Implicit Differentiation, VIII

<u>Example</u>: Suppose that x, y, z satisfy $x^2yz + e^x\cos(y) = 3$.

1. If z is defined implicitly in terms of x and y, find $\partial z/\partial x$.
   - The implicit relation is $f(x, y, z) = 3$ where $f(x, y, z) = x^2yz + e^x\cos(y)$. We then compute $f_x = 2xyz + e^x\cos(y)$, $f_y = x^2z - e^x\sin(y)$, $f_z = x^2y$.
   - By the implicit differentiation formulas, $\dfrac{\partial z}{\partial x} = -\dfrac{\partial f/\partial x}{\partial f/\partial z} = -\dfrac{f_x}{f_z} = -\dfrac{2xyz + e^x\cos(y)}{x^2y}$.
2. If y is defined implicitly in terms of x and z, find $\partial y/\partial z$.
   - Using the calculations above, we get $\dfrac{\partial y}{\partial z} = -\dfrac{\partial f/\partial z}{\partial f/\partial y} = -\dfrac{f_z}{f_y} = -\dfrac{x^2y}{x^2z - e^x\sin(y)}$.

Summary

We introduced the multivariable chain rule and discussed how to apply it.

We discussed implicit differentiation using the multivariable chain rule.

Next lecture: Minima and maxima, classification of critical points.