Math 2321 (Multivariable Calculus)

Lecture #7 of 38 $\sim$ February 3, 2021

Directional Derivatives and Gradient Vectors

- Directional Derivatives
- The Gradient Vector
- Minimum and Maximum Rate of Change

This material represents §2.2.1 from the course notes.

Roadmap

Last time, we developed the notion of the partial derivatives of a function, which measure the rate of change in the coordinate directions.

- Specifically, f_x is the rate of change in the x-direction, f_y is the rate of change in the y-direction, and so forth.

However, we could just as well ask for the rate of change in *any* direction, not just the coordinate directions.

- Our immediate goal is to develop this idea of finding a "directional derivative".
- As it turns out, we can compute these general directional derivatives using the (regular) partial derivatives and an associated vector known as the gradient.

Directional Derivatives, I

So, suppose we have a function $f(x, y)$, a point $P = (p_x, p_y)$, and a unit direction vector $\mathbf{v} = \langle v_x, v_y \rangle$.

- We want to understand the rate of change of f at P as we move in the direction of $\mathbf{v}$.
- To do this, imagine looking at the values of f as we travel along the line through P in the direction of $\mathbf{v}$: we can then just find the rate of change at P as we travel along this line.
- The line is parametrized as $\langle x, y \rangle = \langle p_x, p_y \rangle + t\langle v_x, v_y \rangle$, so $x = p_x + tv_x$ and $y = p_y + tv_y$. Note P corresponds to $t = 0$.
- We then want to find the rate of change of the single-variable function $f(x(t), y(t)) = f(p_x + tv_x, p_y + tv_y)$ at $t = 0$.
- This rate of change is given by the difference quotient $\lim_{h \to 0} \dfrac{f(p_x + hv_x, p_y + hv_y) - f(p_x, p_y)}{h}$.

Directional Derivatives, II

Here is the formal definition of the directional derivative:

Definition

If $\mathbf{v} = \langle v_x, v_y \rangle$ *is a unit vector, then the* <u>*directional derivative of* $f(x, y)$ *in the direction of* $\mathbf{v}$</u> *at* (x, y), *denoted* $D_{\mathbf{v}}(f)(x, y)$, *is defined to be the limit* $D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \dfrac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y)}{h}$, *provided that the limit exists.*

<u>Important Note</u>: In the definition of directional derivative, the vector $\mathbf{v}$ must be a *unit vector*.

- We sometimes will speak of the directional derivative of a function in the direction of a vector $\mathbf{w}$ whose length is not 1.
- What we mean by this is the directional derivative in the direction of the unit vector $\frac{\mathbf{w}}{||\mathbf{w}||}$ in the same direction as $\mathbf{w}$.

Directional Derivatives, III

If $\mathbf{v} = \langle v_x, v_y \rangle$ is a unit vector, then the directional derivative is
$$D_{\mathbf{v}}(f)(x,y) = \lim_{h \to 0} \frac{f\,(x + h\,v_x,\; y + h\,v_y) - f\,(x,y)}{h}.$$

- The limit in the definition (summarized above) is explicit, but a little bit hard to understand as written.
- If we write things in vector notation, with $\mathbf{x} = \langle x, y \rangle$, then the definition might be clearer to you: it becomes $D_{\mathbf{v}}(f)(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\,\mathbf{v}) - f(\mathbf{x})}{h}$.
- The difference quotient is the ratio $\Delta f / \Delta h$, where Δf is the amount that f changes by moving a distance Δh in the direction of $\mathbf{v}$, as discussed during the motivation earlier.
- Compare it to the definition of the derivative of a function of one variable: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Directional Derivatives, IV

If $\mathbf{v} = \langle v_x, v_y \rangle$ is a unit vector, then the directional derivative is
$$D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \frac{f\,(x + h\,v_x,\ y + h\,v_y) - f\,(x, y)}{h}.$$
When $\mathbf{v}$ is the unit vector in one of the coordinate directions, the directional derivative reduces to the corresponding partial derivative. Explicitly:

- If $\mathbf{v} = \langle 1, 0 \rangle$, the unit vector in the x-direction, then $$D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \frac{f(x + h,\ y) - f(x, y)}{h} = f_x(x, y).$$
- If $\mathbf{v} = \langle 0, 1 \rangle$, the unit vector in the y-direction, then $$D_{\mathbf{v}}(f)(x, y) = \lim_{h \to 0} \frac{f(x,\ y + h) - f(x, y)}{h} = f_y(x, y).$$

Directional Derivatives, V

Computing directional derivatives using the limit definition is generally quite tedious.

Example: If $\mathbf{v} = \langle \frac{3}{5}, \frac{4}{5} \rangle$, and $f(x, y) = 2x + y$, find the directional derivative of f in the direction of $\mathbf{v}$ at $(x, y) = (1, 6)$.

Directional Derivatives, V

Computing directional derivatives using the limit definition is generally quite tedious.

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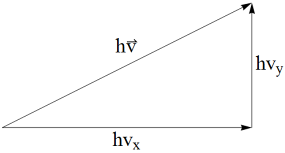
Since $\mathbf{v}$ is a unit vector, the definition says that

$$
\begin{aligned}
D_{\mathbf{v}}(f)(1,6) &= \lim_{h \to 0} \frac{f(x + h\,v_x,\ y + h\,v_y) - f(x, y)}{h} \\
&= \lim_{h \to 0} \frac{f(1 + \frac{3}{5}h,\ 6 + \frac{4}{5}h) - f(1,6)}{h} \\
&= \lim_{h \to 0} \frac{\left[2(1 + \frac{3}{5}h) + (6 + \frac{4}{5}h)\right] - [2 \cdot 1 + 6]}{h} \\
&= \lim_{h \to 0} \frac{(8 + 2h) - 8}{h} = 2.
\end{aligned}
$$

Directional Derivatives, VI

We now describe an easier way to compute directional derivatives:

- We want to understand the behavior of $f(\mathbf{x} + h\mathbf{v})$ as $h \to 0$.
- We can break $h\mathbf{v}$ into horizontal and vertical components:



- Then the total change in f is the sum of the horizontal and vertical changes.

Directional Derivatives, VI

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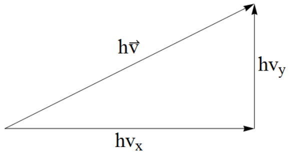

- Then the total change in f is the sum of the horizontal and vertical changes.
- The horizontal change is determined by the x-partial f_x, while the vertical change is determined by the y-partial f_y.
- Specifically, the horizontal change is $f_x \cdot hv_x$, while the vertical change is $f_y \cdot hv_y$. Therefore, the total change is $f_x \cdot hv_x + f_y \cdot hv_y$, so if we divide by h and take the limit, we get the simple expression $f_x v_x + f_y v_y$.

The Gradient Vector, I

We can package this last calculation using vector language:

Definition

The gradient of a function $f(x, y)$, denoted ∇f, is the vector-valued function $\nabla f(x, y) = \langle f_x(x, y),\ f_y(x, y)\rangle$.
For a function $g(x, y, z)$, the gradient ∇g is $\nabla g(x, y, z) = \langle g_x(x, y, z),\ g_y(x, y, z),\ g_z(x, y, z)\rangle$.

- Note: The symbol ∇ is called "nabla", and is pronounced either as "nabla" or as "del".
- If f is a function of some other number of variables, the gradient is defined analogously.
- Note that the gradient of f is a vector-valued function: it takes the same number of arguments as f does, and outputs a vector in the same number of coordinates.

The Gradient Vector, II

Example: Find the gradient vector for each given function:

1. $f(x, y) = x^2 \cos(y)$.
2. $g(x, y, z) = x^2 + y^2 + z^2$.
3. $p(x, y, z) = \sin(yz^2)$.

The Gradient Vector, II

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- We have $\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle 2x \cos(y), -x^2 \sin(y) \right\rangle$.

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- We have $\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle 2x \cos(y), -x^2 \sin(y) \right\rangle$.
- We have $\nabla g(x, y, z) = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, 2z \rangle$.

The Gradient Vector, II

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- We have $\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle 2x\ \cos(y),\ -x^2 \sin(y) \right\rangle$.
- We have $\nabla g(x, y, z) = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, 2z \rangle$.
- We have
  $\nabla p(x, y, z) = \langle p_x, p_y, g_z \rangle = \left\langle 0, z^2 \cos(yz^2), 2yz \cos(yz^2) \right\rangle$.

The Gradient Vector, II

Our main result is that we can use the gradient vector to compute directional derivatives:

Theorem (Gradient and Directional Derivatives)

If $\mathbf{v}$ *is any unit vector, and* f *is a function all of whose partial derivatives are continuous, then the directional derivative* $D_{\mathbf{v}}f$ *satisfies* $D_{\mathbf{v}}f = \nabla f \cdot \mathbf{v}$*. In words, the directional derivative of* f *in the direction of* $\mathbf{v}$ *is the dot product of* ∇f *with the vector* $\mathbf{v}$*.*

Warning: This result requires the direction $\mathbf{v}$ to be a unit vector. If the desired direction is not a unit vector, it is necessary to normalize the direction vector first! (Otherwise, the formula will give the wrong answer.)

The Gradient Vector, III

The proof of the theorem is really just a formalization of the geometric argument we gave before, about breaking the direction vector **v** into its components in the coordinate directions and then adding up the changes in the function from each piece.

On the next slides I will give the algebraic details for the two-variable case, which mostly involve wrestling with the limit definition.

If $v_x = 0$ then the directional derivative is the x-partial (or its negative), and if $v_y = 0$ then the directional derivative is the y-partial (or its negative), and the result is true. So the only case of interest is when v_x and v_y are both nonzero.

The Gradient Vector, IV

Proof:

- If v_x and v_y are both nonzero, then $D_{\mathbf{v}}(f)(x,y)$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x+h\,v_x,\, y+h\,v_y)-f(x,y)}{h}, \\
&= \lim_{h\to 0} \left[\frac{f(x+h\,v_x,\, y+h\,v_y)-f(x,y+h\,v_y)}{h} + \frac{f(x,y+h\,v_y)-f(x,y)}{h}\right] \\
&= \lim_{h\to 0} \left[\frac{f(x+h\,v_x,\, y+h\,v_y)-f(x,y+h\,v_y)}{h\,v_x}\,v_x + \frac{f(x,y+h\,v_y)-f(x,y)}{h\,v_y}\,v_y\right] \\
&= v_x \lim_{h\to 0} \left[\frac{f(x+h\,v_x,\, y+h\,v_y)-f(x,y+h\,v_y)}{h\,v_x}\right] + v_y \lim_{h\to 0} \left[\frac{f(x,y+h\,v_y)-f(x,y)}{h\,v_y}\right] \\
&= v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} = \langle v_x, v_y\rangle \cdot \langle f_x, f_y\rangle = \mathbf{v}\cdot\nabla f \text{ (as claimed)},
\end{aligned}
$$

where we used the continuity of f_x to evaluate the first term and the definition of f_y to evaluate the second term.

Gradients and Directional Derivatives, I

Here is the same directional derivative example as before:

Example: If $\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, and $f(x, y) = 2x + y$, find the directional derivative of f in the direction of $\mathbf{v}$ at $(1, 6)$.

Gradients and Directional Derivatives, I

Here is the same directional derivative example as before:

Example: If $\mathbf{v} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, and $f(x, y) = 2x + y$, find the directional derivative of f in the direction of $\mathbf{v}$ at $(1, 6)$.

- We compute the gradient: $f_x = 2$ and $f_y = 1$, so $\nabla f = \langle 2, 1 \rangle$.
- Then $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = \langle 2, 1 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{6}{5} + \frac{4}{5} = 2$.

Observe how much easier it was to use the gradient to compute the directional derivative! (No limits and very little computation.)

Gradients and Directional Derivatives, II

Example: Find the rate of change of the function $f(x, y, z) = x^2 z + y^3$ at the point $(x, y, z) = (1, -1, 2)$ in the direction of the unit vector $\mathbf{w} = \frac{1}{13} \langle 3, 12, 4 \rangle$.

Gradients and Directional Derivatives, II

Example: Find the rate of change of the function $f(x, y, z) = x^2 z + y^3$ at the point $(x, y, z) = (1, -1, 2)$ in the direction of the unit vector $\mathbf{w} = \frac{1}{13}\langle 3, 12, 4\rangle$.

- Note that $||\mathbf{w}|| = 1$ is a unit vector, so we can use the theorem directly.
- We compute $\nabla f = \left\langle 2xz, 3y^2, x^2\right\rangle$, so $\nabla f(1, -1, 2) = \langle 4, 3, 1\rangle$.
- Then the desired rate of change is $D_{\mathbf{w}} f = \nabla f \cdot \mathbf{w} = \frac{3}{13} \cdot 4 + \frac{12}{13} \cdot 3 + \frac{4}{13} \cdot 1 = \frac{52}{13} = 4$.

Gradients and Directional Derivatives, III

<u>Example</u>: Find the rate of change of the function $f(x, y, z) = e^{xyz}$ at the point $(x, y, z) = (1, 1, 1)$ in the direction of the vector $\mathbf{w} = \langle -2, 1, 2 \rangle$.

Gradients and Directional Derivatives, III

Example: Find the rate of change of the function $f(x, y, z) = e^{xyz}$ at the point $(x, y, z) = (1, 1, 1)$ in the direction of the vector $\mathbf{w} = \langle -2, 1, 2 \rangle$.

- Note that $\mathbf{w}$ is not a unit vector, so we must normalize it: since $||\mathbf{w}|| = \sqrt{(-2)^2 + 1^2 + 2^2} = 3$, we take $\mathbf{v} = \dfrac{\mathbf{w}}{||\mathbf{w}||} = \left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{2}{3} \right\rangle$.
- Now we compute $\nabla f = \langle yz\, e^{xyz},\ xz\, e^{xyz},\ xy\, e^{xyz} \rangle$, so $\nabla f(1, 1, 1) = \langle e, e, e \rangle$.
- Then the desired rate of change is $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = -\dfrac{2}{3}e + \dfrac{1}{3}e + \dfrac{2}{3}e = \dfrac{1}{3}e \approx 0.9061.$

Gradients and Directional Derivatives, IV

Example: Find the rate of change of the function $f(x, y) = \ln(x^2 + y^2)$ at the point $(1, 2)$ in the direction towards the point $(3, 3)$.

Gradients and Directional Derivatives, IV

Example: Find the rate of change of the function $f(x, y) = \ln(x^2 + y^2)$ at the point $(1, 2)$ in the direction towards the point $(3, 3)$.

- Here, our direction vector is $\mathbf{w} = \langle 3, 3\rangle - \langle 1, 2\rangle = \langle 2, 1\rangle$, which we must normalize since it is not a unit vector.
- The normalization is $\mathbf{v} = \frac{1}{\sqrt{5}}\langle 2, 1\rangle = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$.
- Then $\nabla f = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$, so $\nabla f(1, 2) = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$.
- Therefore, the desired rate of change is $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} = \frac{2}{\sqrt{5}} \cdot \frac{2}{5} + \frac{1}{\sqrt{5}} \cdot \frac{4}{5} = \frac{8}{5\sqrt{5}} \approx 0.7155.$

Gradients and Directional Derivatives, V

From the gradient theorem for computing directional derivatives, we can deduce several corollaries about how the magnitude of the directional derivative depends on the direction $\mathbf{v}$:

Corollary (Minimum and Maximum Increase)

Suppose f is a differentiable function with gradient $\nabla f \neq \mathbf{0}$ and $\mathbf{v}$ is a unit vector. Then the following hold:

1. *The maximum value of $D_{\mathbf{v}}f$ occurs when $\mathbf{v}$ is a unit vector in the direction of ∇f, and the maximum value is $||\nabla f||$.*
2. *The minimum value of $D_{\mathbf{v}}f$ occurs when $\mathbf{v}$ is a unit vector in the opposite direction of ∇f, and the minimum is $-||\nabla f||$.*
3. *The value of $D_{\mathbf{v}}f$ is zero if and only if $\mathbf{v}$ is orthogonal to the gradient ∇f.*

Gradients and Directional Derivatives, VI

To summarize these results:

- The direction where f is increasing most rapidly is the direction of the gradient ∇f.
- The direction where f is decreasing most rapidly is the opposite direction of the gradient $-\nabla f$.
- The maximum rate of increase or decrease is the length of the gradient vector.
- The value of f is not changing in any direction orthogonal to ∇f.

The idea of the proof is simply to use the dot product theorem to analyze the expression $\nabla f \cdot \mathbf{v}$.

Gradients and Directional Derivatives, VII

Proof:

- If $\mathbf{v}$ is a unit vector, then the directional derivative satisfies $D_{\mathbf{v}}f = \nabla f \cdot \mathbf{v} = ||\nabla f|| \, ||\mathbf{v}|| \cos(\theta)$, where θ is the angle between ∇f and $\mathbf{v}$.
- We know that $||\nabla f||$ is a fixed nonnegative number, and $||\mathbf{v}|| = 1$. So if we change the direction of $\mathbf{v}$, the only quantity in $||\nabla f|| \, ||\mathbf{v}|| \cos(\theta)$ that changes is $\cos(\theta)$.
- So, the maximum value of $\nabla f \cdot \mathbf{v}$ occurs when $\cos(\theta) = 1$, which is to say, when ∇f and $\mathbf{v}$ are parallel and point in the same direction. The maximum value is then just $||\nabla f||$.
- The minimum value of $\nabla f \cdot \mathbf{v}$ occurs when $\cos(\theta) = -1$, which is to say, when ∇f and $\mathbf{v}$ are parallel and point in opposite directions. The minimum value is then just $-||\nabla f||$.
- Finally $D_v f$ is zero if and only if $\nabla f \cdot \mathbf{v} = 0$, which is equivalent to saying that ∇f and $\mathbf{v}$ are orthogonal.

Gradients and Directional Derivatives, VIII

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

1. In what direction is f increasing fastest at P, and how fast?
2. In what direction is f decreasing fastest at P, and how fast?
3. Find a direction in which f is not changing at P.

Gradients and Directional Derivatives, VIII

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

1. In what direction is f increasing fastest at P, and how fast?
2. In what direction is f decreasing fastest at P, and how fast?
3. Find a direction in which f is not changing at P.

- The function is increasing the fastest in the direction of the gradient and decreasing the fastest in the direction opposite to the gradient, and the corresponding maximum rate of increase (or decrease) is the magnitude of the gradient.
- We have $f_x = 2x$ and $f_y = 2y$ so $\nabla f = \langle 2x, 2y \rangle$, and $\nabla f(3, 4) = \langle 6, 8 \rangle$.

Gradients and Directional Derivatives, IX

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

1. In what direction is f increasing fastest at P, and how fast?

Gradients and Directional Derivatives, IX

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

1. In what direction is f increasing fastest at P, and how fast?
   - Since $||\nabla f(3, 4)|| = \sqrt{6^2 + 8^2} = 10$, the maximum value of the directional derivative $D_{\mathbf{v}} f$ is 10 and occurs in the direction of $\dfrac{\langle 6, 8 \rangle}{10} = \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$.
2. In what direction is f decreasing fastest at P, and how fast?

Gradients and Directional Derivatives, IX

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

1. In what direction is f increasing fastest at P, and how fast?
   - Since $||\nabla f(3, 4)|| = \sqrt{6^2 + 8^2} = 10$, the maximum value of the directional derivative $D_{\mathbf{v}} f$ is 10 and occurs in the direction of $\dfrac{\langle 6, 8\rangle}{10} = \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$.
2. In what direction is f decreasing fastest at P, and how fast?
   - Likewise, the minimum value of $D_{\mathbf{v}} f$ is -10 and occurs in the direction of $-\dfrac{\langle 6, 8\rangle}{10} = \left\langle -\dfrac{3}{5}, -\dfrac{4}{5} \right\rangle$.

Gradients and Directional Derivatives, X

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

3. Find a direction in which f is not changing at P.

Gradients and Directional Derivatives, X

Example: Let $f(x, y) = x^2 + y^2$ and take $P = (3, 4)$.

3. Find a direction in which f is not changing at P.

- For this, we need to find a direction vector that is orthogonal to $\nabla f(3, 4) = \langle 6, 8 \rangle$.
- If we explicitly write $\mathbf{v} = \langle v_x, v_y \rangle$, then we must have $6v_x + 8v_y = 0$ and $v_x^2 + v_y^2 = 1$.
- The first equation gives $v_y = (-3/4)v_x$ and then the second equation yields $v_x^2 + (9/16)v_x^2 = 1$ so that $v_x^2 = 16/25$ and so $v_x = \pm 4/5$.
- So there are two possible direction vectors: either $\mathbf{v} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$ or its negative $\left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$.

Gradients and Directional Derivatives, XI

Example: For $f(x, y, z) = x^3 + y^3 + 2z^3$, in which direction is f increasing the fastest at $(x, y, z) = (2, -2, 1)$, and how fast? In which direction is f decreasing the fastest, and how fast?

Gradients and Directional Derivatives, XI

<u>Example</u>: For $f(x, y, z) = x^3 + y^3 + 2z^3$, in which direction is f increasing the fastest at $(x, y, z) = (2, -2, 1)$, and how fast? In which direction is f decreasing the fastest, and how fast?

- Note f is increasing the fastest in the direction of ∇f and decreasing the fastest in the direction opposite ∇f, and the corresponding maximum rate of increase/decrease is $||\nabla f||$.
- We have $\nabla f(x, y, z) = \left\langle 3x^2, 3y^2, 6z^2 \right\rangle$, so $\nabla f(2, -2, 1) = \langle 12, 12, 6 \rangle$.
- Since $||\nabla f(2, -2, 1)|| = \sqrt{12^2 + 12^2 + 6^2} = 18$, we see that the maximum value of the directional derivative $D_{\mathbf{v}} f$ is 18 and occurs in the direction of $\frac{\langle 12,12,6 \rangle}{18} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$.
- Likewise, the minimum value of $D_{\mathbf{v}} f$ is -18 and occurs in the direction of $-\frac{\langle 12,12,6 \rangle}{18} = \left\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \right\rangle$.

Geometry of Directional Derivatives, I

We can also use level sets to visualize directional derivatives, like with partial derivatives.

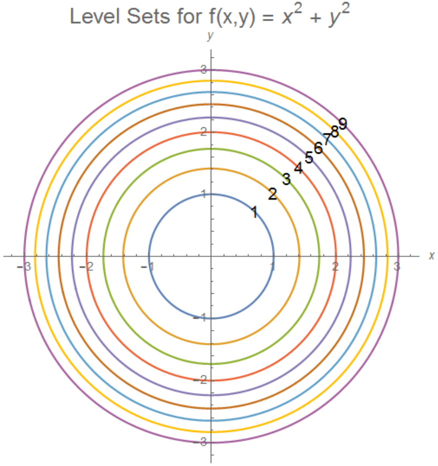
- If we draw the level curves for a function $f(x, y)$, we can estimate the value of the directional derivative $D_{\mathbf{v}} f(P)$ by looking at the behavior of f in the **v**-direction near P.
- If moving along **v** from P crosses over level curves corresponding to larger values of f, then $D_{\mathbf{v}} f(P) > 0$.
- Inversely, if moving along **v** from P crosses over level curves corresponding to smaller values of f, then $D_{\mathbf{v}} f(P) < 0$.

We can also estimate the approximate value of $D_{\mathbf{v}} f(P)$ based on how much f changes as we move in the x-direction:

- Per the limit definition, if the value of f changes by a total amount Δf as we move a distance Δh in the **v**-direction from P, then $D_{\mathbf{v}} f(P)) \approx \Delta f / \Delta h$.

Geometry of Directional Derivatives, II

For example, here are the level sets for $f(x, y) = x^2 + y^2$:



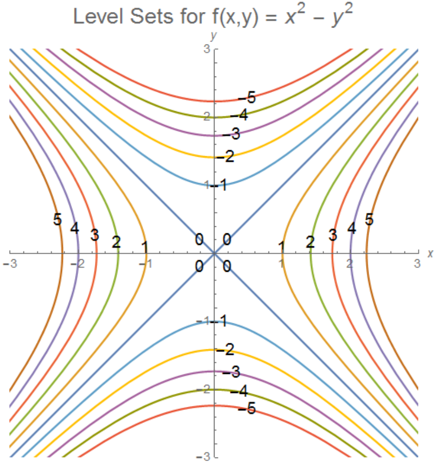
Consider $P = (1, 1)$, located on the level set where $f = 2$.

- If we move along $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$, we move towards the level sets where $f = 3, 4, 5, \ldots$. This means $D_{\mathbf{v}} f(P) > 0$.

- If we move along $\mathbf{v} = \langle -1, 0\rangle$, we move towards the level set where $f = 1$, so $D_{\mathbf{v}} f(P) < 0$.

Geometry of Directional Derivatives, III

For another example, here are the level sets for $f(x, y) = x^2 - y^2$:



Consider $P = (1, 1)$, located on the level set where $f = 0$.

- If we move along $\mathbf{v} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$, we stay on the level set with $f = 0$. This means $D_{\mathbf{v}} f(P) = 0$.
- If we move along $\mathbf{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle$, we move towards the level set where $f = -1$, so $D_{\mathbf{v}} f(P) < 0$.
- If we move along $\mathbf{v} = \frac{1}{\sqrt{5}}\langle 2, 1\rangle$, we move towards the level set where $f = 1$, so $D_{\mathbf{v}} f(P) > 0$.

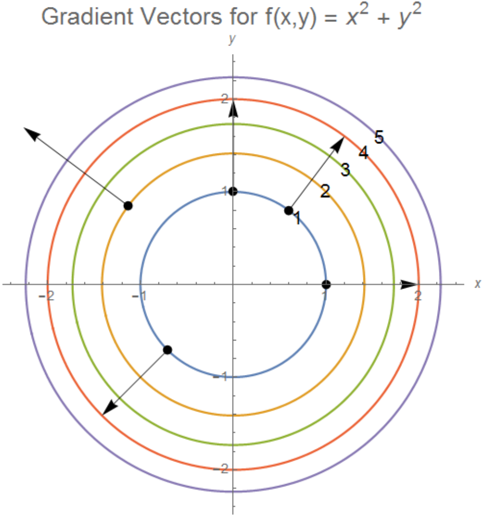
Geometry of the Gradient, I

We can also view the gradient vector ∇f geometrically.

- Specifically, as we just discussed, ∇f points in the direction of fastest increase for f.
- When we consider the plot of level sets, what this means is that ∇f will be (more or less) pointing directly towards the nearest level curve to P where f takes a larger value.
- Intuitively, this should make sense: the fastest way to climb a hill is to follow the steepest incline; inversely, the fastest way to go down is to follow the steepest descent.

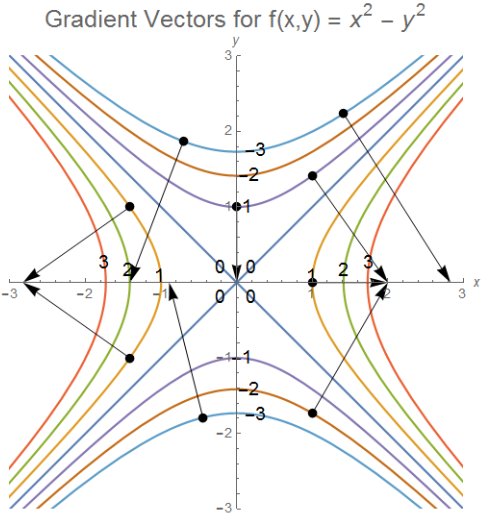
Geometry of the Gradient, II

Here are plots of some gradient vectors for $f(x, y) = x^2 + y^2$:



Geometry of the Gradient, III

Here are plots of some gradient vectors for $f(x, y) = x^2 - y^2$:



Summary

We introduced directional derivatives and gradient vectors and discussed how to compute and interpret them.

We discussed how to find the minimum and maximum rate of change of a function.

Next lecture: Tangent lines and planes, linearization.