Math 2321 (Multivariable Calculus)

Lecture #6 of 38 $\sim$ February 1, 2021

Limits and Partial Derivatives

- Limits
- Partial Derivatives

This material represents §2.1 from the course notes.

Overview of §2: Partial Derivatives

We now move into the second chapter of the course, where we will discuss how to generalize the idea of a derivative to functions of several variables.

- Today, we will have a brief discussion of limits and continuity (since the derivatives we discuss will all be defined in terms of limits) and introduce partial derivatives, which capture the notion of the rate of change of a function in one of the coordinate directions.
- Next, we will generalize the idea of a partial derivative to that of a directional derivative, which measures the rate of change in an arbitrary direction.
- The rest of the chapter is devoted to other familiar topics, all related to derivatives: tangent lines and planes, the chain rule, linearization, classification of critical points (minima and maxima), and various types of optimization problems.

Limits (As Quickly As Possible), I

Here is the official definition of limit for a function of 2 variables:

Definition

A function $f(x, y)$ has the limit L as $(x, y) \to (a, b)$, written as $\lim_{(x,y)\to(a,b)} f(x, y) = L$, if, for any $\epsilon > 0$ there exists a $\delta > 0$ with the property that for all (x, y) with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have that $|f(x, y) - L| < \epsilon$.

Roughly speaking, it works as follows:

- Suppose you claim f has a limit L as $(x, y) \to P$.
- In order to convince me that the function really does have that limit, I challenge you by handing you some small value $\epsilon > 0$, and I want you to give me some value of δ, with the property that $f(x)$ is always within ϵ of the limit value L, for all the points that are within a distance δ of P.
- If you can always meet the challenge, no matter what ϵ I give you, then I agree the limit really is L.

Limits (As Quickly As Possible), II

The definition of limit is not easy to work with, and we won't work with it. Instead, one uses the definition to establish various basic limit evaluations.

- For example, one can show $\lim_{(x,y)\to(a,b)} c = c$ for any constant c, and also $\lim_{(x,y)\to(a,b)} x = a$, $\lim_{(x,y)\to(a,b)} y = b$.

Likewise, one obtains all of the usual limit laws:

- Explicitly, suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L_f$ and $\lim_{(x,y)\to(a,b)} g(x,y) = L_g$.
- Then $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L_f + L_g$, and $\lim_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] = L_f - L_g$, and $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = L_f L_g$, and $\lim_{(x,y)\to(a,b)} [f(x,y)/g(x,y)] = L_f/L_g$ provided $L_g \neq 0$, and so forth....

Limits (As Quickly As Possible), III

We say a function is <u>continuous</u> if it equals its limit. Using the limit rules and basic limit evaluations, we can establish that the usual slate of functions is continuous:

- Any polynomial in x and y is continuous everywhere.
- Any quotient of polynomials $\frac{p(x,y)}{q(x,y)}$ is continuous everywhere that the denominator is nonzero.
- The exponential, sine, and cosine of any continuous function are all continuous everywhere.
- The logarithm of a positive continuous function is continuous.

Limits (As Quickly As Possible), IV

For one-variable limits, we also have a notion of "one-sided" limits, namely, the limits that approach the target point either from above or from below.

- In the multiple-variable case, there are many more paths along which we can approach our target point.
- For example, if our target point is the origin $(0, 0)$, then we could approach along the positive x-axis, or the positive y-axis, or along any line through the origin... or along the curve $y = x^2$, or any other continuous curve that passes through $(0, 0)$.

Limits (As Quickly As Possible), V

As with limits in one variable, if limits from different directions have different values, then the overall limit does not exist:

Proposition ("Two Paths Test")

Let $f(x,y)$ be a function of two variables and (a,b) be a point. If there are two continuous paths passing through the point (a,b) such that f has different limits as $(x,y)\to(a,b)$ along the two paths, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

The proof is essentially just the definition of limit: if the limit exists, then f must stay within a very small range near (a,b). But if there are two paths through (a,b) along which f approaches different values, then the values of f near (a,b) do not stay within a small range, so the limit cannot exist.

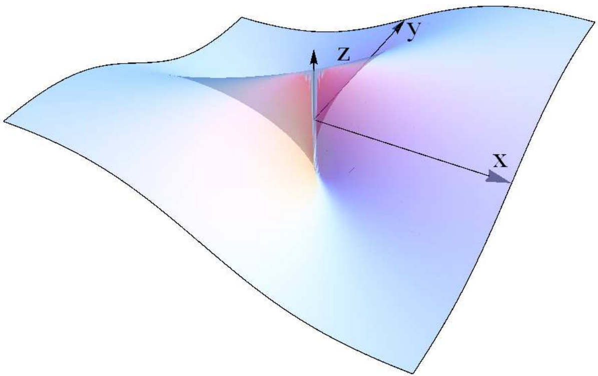
Limits (As Quickly As Possible), VI

Example: Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

- We try some simple paths: along the path $(x, y) = (t, 0)$ as $t \to 0$ the limit becomes $\lim_{t\to 0} \dfrac{t \cdot 0}{t^2 + 0^2} = \lim_{t\to 0} 0 = 0$.
- Along the path $(x, y) = (0, t)$ as $t \to 0$ we have $\lim_{t\to 0} \dfrac{0 \cdot t}{0^2 + t^2} = \lim_{t\to 0} 0 = 0$.
- Along these two paths the limits are equal. But this does not show the existence of the limit.
- Let's try along the path $(x, y) = (t, t)$: the limit then becomes $\lim_{t\to 0} \dfrac{t^2}{t^2 + t^2} = \lim_{t\to 0} \dfrac{1}{2} = \dfrac{1}{2}$.
- Thus, along the two paths $(x, y) = (0, t)$ and $(x, y) = (t, t)$, the function has different limits as $(x, y) \to (0, 0)$.
- Hence the limit does not exist.

Limits (As Quickly As Possible), VII

Here is a plot of $z = (xy)/(x^2 + y^2)$ near $(0, 0)$:



Limits (As Quickly As Possible), VIII

Example: Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4 + y^2}$ does not exist.

- If we try along the line $x = 0$, with $(x, y) = (0, t)$, then the limit becomes $\lim_{t\to 0} \dfrac{0}{t^2} = 0$.
- If we try along the line $y = mx$, with $(x, y) = (t, mt)$, then the limit becomes $\lim_{t\to 0} \dfrac{mt^3}{t^2 + t^4} = \lim_{t\to 0} \dfrac{mt}{1 + t^2} = 0$.
- So it seems like the limit might be zero, since it is zero along any line approaching the origin.

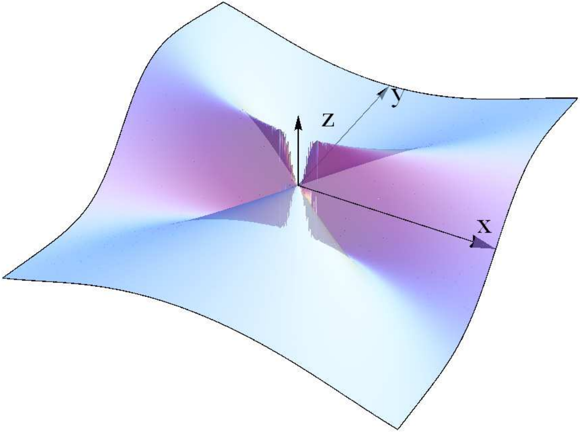
Limits (As Quickly As Possible), VIII

Example: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist.

- If we try along the line $x = 0$, with $(x, y) = (0, t)$, then the limit becomes $\lim_{t\to 0} \frac{0}{t^2} = 0$.
- If we try along the line $y = mx$, with $(x, y) = (t, mt)$, then the limit becomes $\lim_{t\to 0} \frac{mt^3}{t^2+t^4} = \lim_{t\to 0} \frac{mt}{1+t^2} = 0$.
- So it seems like the limit might be zero, since it is zero along any line approaching the origin.
- But, quite strangely, the limit actually does not exist! If we go along the parabola $y = x^2$, with $(x, y) = (t, t^2)$, then the limit becomes $\lim_{t\to 0} \frac{t^4}{t^4+t^4} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$.
- So this limit actually does not exist.

Limits (As Quickly As Possible), IX

Here is a plot of $z = (x^2 y)/(x^4 + y^2)$ near $(0, 0)$:



Partial Derivatives, I

Now that we have very briefly discussed limits, we can get to the main attraction: partial derivatives.

- Imagine we wanted to try to give an answer to the question "What is the derivative of $f(x, y) = x^2 + y^2$?".
- One option would be to think back to the original definition of derivative, as the rate of change of a function: we can ask "if we change x by some small amount, how much does f change by?". (And we can formalize that question using limits.)
- However, there is no reason to ask this question only about x!
- The function f also depends on the variable y, so we could just as well ask about what happens to values of f as we change y.
- Since both of these questions are very reasonable things to ask, we will just ask both of them!

Partial Derivatives, II

We first define the "partial derivative with respect to x":

Definition

For a function $f(x, y)$ of two variables, we define the <u>*partial derivative*</u> *of f with respect to x as*

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Observe that the numerator of $\partial f / \partial x$ is evaluating how much the value of f is changing if we change x to $x + h$, but y is left unchanged.

The difference quotient $\partial f / \partial x$ is therefore measuring how fast f is changing as we change the value of x, but keep y fixed.

Partial Derivatives, III

We also have an analogous "partial derivative with respect to y":

Definition

For a function $f(x, y)$ of two variables, we define the *<u>partial derivative</u>* *of f with respect to y as*

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Notice now that $\partial f / \partial y$ is measuring how fast f is changing as we change the value of y, but keep x fixed.

Partial Derivatives, IV

A few notational / terminological remarks:

- In multivariable calculus, we use the symbol ∂ (typically pronounced either like the letter d or as "del") to denote taking a derivative, in contrast to single-variable calculus where we use the symbol d.
- We will frequently use both notations $\dfrac{\partial f}{\partial y}$ and f_y to denote partial derivatives: I generally use the difference quotient notation to emphasize a formal property of a derivative, and the subscript notation when I want to save space.
- Do not use the one-variable "prime" notation (f') with functions of more than one variable, because it is not clear which variable the function is being differentiated with respect to.

Partial Derivatives, V

Although partial derivatives are defined in terms of limits, we can in fact use all of our usual differentiation rules to compute them.

- Specifically, to evaluate a partial derivative of the function f with respect to x, we need only pretend that all the other variables (i.e., everything except x) that f depends on are constants.
- Then we just evaluate the derivative of f with respect to x like a normal one-variable derivative.
- And, of course, the differentiation rules (the product rule, quotient rule, chain rule, etc.) from one-variable calculus still hold: there will just be extra variables floating around.

Partial Derivatives, VI

Example: Find f_x and f_y for $f(x, y) = x^3y^2 + e^x$.

- For f_x, we treat y as a constant and x as the variable.
- Thus, $f_x = 3x^2 \cdot y^2 + e^x$.
- Similarly, to find f_y, we instead treat x as a constant and y as the variable.
- Thus, $f_y = x^3 \cdot 2y + 0 = 2x^3y$. (Note in particular that the derivative of e^x with respect to y is zero.)

Partial Derivatives, VII

Example: Find f_x and f_y for $f(x, y) = \ln(x^3 + y^4)$.

Partial Derivatives, VII

Example: Find f_x and f_y for $f(x, y) = \ln(x^3 + y^4)$.

- For f_x, we treat y as a constant and x as the variable.
- We can apply the chain rule to get $f_x = \dfrac{3x^2}{x^3 + y^4}$, since the derivative of the inner function $x^3 + y^4$ with respect to x is $3x^2$. (Remember that y is constant, so its x-derivative is zero.)
- Similarly, we can use the chain rule to find the partial derivative $f_y = \dfrac{4y^3}{x^3 + y^4}$.

Partial Derivatives, VIII

Example: Find f_x and f_y for $f(x, y) = \dfrac{e^{xy}}{x^2 + x}$.

Partial Derivatives, VIII

Example: Find f_x and f_y for $f(x,y) = \dfrac{e^{xy}}{x^2+x}$.

- For f_x we apply the quotient rule:
  $f_x = \dfrac{\frac{\partial}{\partial x}\left[e^{xy}\right] \cdot (x^2+x) - e^{xy} \cdot \frac{\partial}{\partial x}\left[x^2+x\right]}{(x^2+x)^2}$.
- Then we can evaluate the derivatives in the numerator to get
  $f_x = \dfrac{(y\,e^{xy}) \cdot (x^2+x) - e^{xy} \cdot (2x+1)}{(x^2+x)^2}$.
- For f_y, the calculation is easier because the denominator is not a function of y.
- So in this case, we just need to use the chain rule to get
  $f_y = \dfrac{1}{x^2+x} \cdot (x\,e^{xy})$.

Partial Derivatives, IX

We can generalize partial derivatives to functions of more than two variables, in the natural way.

- For each input variable, we get a partial derivative with respect to that variable.
- Thus, for example, a function $f(x, y, z)$ would have three different partial derivatives: f_x, f_y, and f_z.
- To evaluate a partial derivative, treat all variables except the variable of interest as constants, and then differentiate with respect to the variable of interest.

Partial Derivatives, X

Example: Find f_x, f_y, and f_z for $f(x, y, z) = x^2y + x^3yz$.

Partial Derivatives, X

<u>Example</u>: Find f_x, f_y, and f_z for $f(x, y, z) = x^2 y + x^3 yz$.

- For f_x we think of y and z as constants.
- Thus, $f_x = (2x)y + (3x^2)yz$.
- For f_y we think of x and z as constants.
- Thus, $f_y = x^2 + x^3 z$.
- For f_z we think of x and y as constants.
- Thus, $f_z = 0 + x^3 y = x^3 y$.

Partial Derivatives, XI

Example: Find f_x, f_y, and f_z for $f(x, y, z) = y\,z\,e^{2x^2 - y}$.

Partial Derivatives, XI

Example: Find f_x, f_y, and f_z for $f(x,y,z) = y\,z\,e^{2x^2-y}$.

- By the chain rule we have $f_x = y\,z \cdot e^{2x^2-y} \cdot 4x$. (We don't need the product rule for f_x since y and z are constants.)
- For f_y we need to use the product rule since f is a product of two nonconstant functions of y.
- We get $f_y = z \cdot e^{2x^2-y} + y\,z \cdot \frac{\partial}{\partial y}\left[e^{2x^2-y}\right]$, and then using the chain rule gives $f_y = z\,e^{2x^2-y} - y\,z \cdot e^{2x^2-y}$.
- For f_z, all of the terms except for z are constants, so we have $f_z = y\,e^{2x^2-y}$.

Partial Derivatives, XII

Example: Find $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$ for $g(s,t) = \sqrt{s^2 + 4st^2}$.

Partial Derivatives, XII

Example: Find $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$ for $g(s,t) = \sqrt{s^2 + 4st^2}$.

- We get $\frac{\partial g}{\partial s} = g_s = \frac{1}{2}(s^2 + 4st^2)^{-1/2} \cdot (2s + 4t^2)$ by the chain rule.
- We get $\frac{\partial g}{\partial t} = g_t = \frac{1}{2}(s^2 + 4st^2)^{-1/2} \cdot (8st)$ by the chain rule.

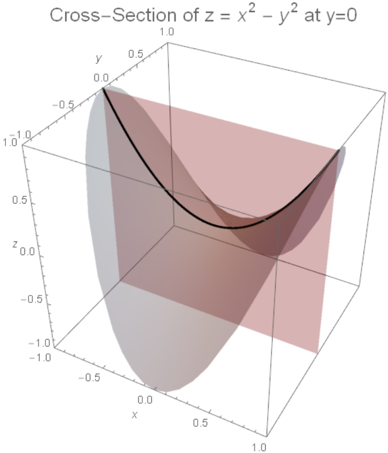
Geometry of Partial Derivatives, I

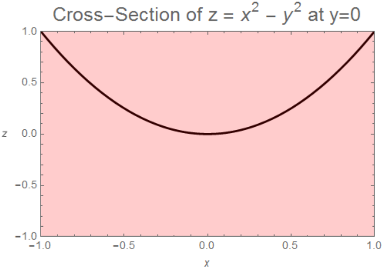
We can also interpret the partial derivatives in various geometric ways.

- Per the definition, for a function $f(x, y)$ of two variables, the partial derivative f_x represents the rate of change of f as x changes but y is held constant.
- Therefore, if we look at the vertical cross-section of the surface $z = f(x, y)$ when $y = b$, the slope of the tangent line to the resulting curve at the point $x = a$ is the value of the partial derivative $f_x(a, b)$.

Geometry of Partial Derivatives, II

Here is the cross-section at $y = 0$ of the surface $z = x^2 - y^2$:

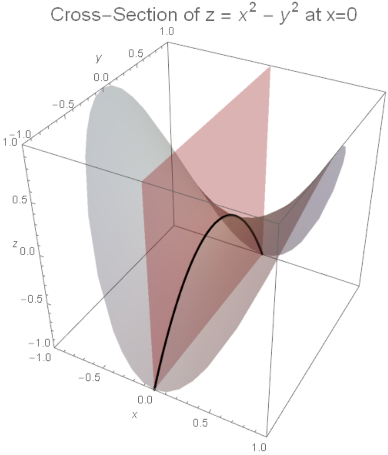


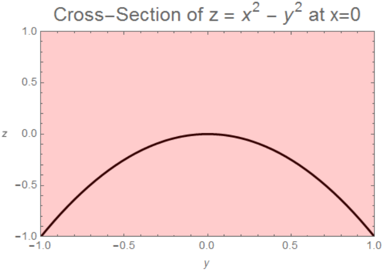


Here, f_x measures the slope of the tangent line to the curve.

Geometry of Partial Derivatives, III

In the same way, f_y represents the slope of the tangent line in a cross-section of $z = f(x, y)$ where x is held constant:





Geometry of Partial Derivatives, IV

We can also use level sets to visualize the partial derivatives.

- If we draw the level curves for a function $f(x, y)$, then we can estimate the value of the partial derivative f_x at a given point $P = (a, b)$ by looking at the behavior of the function in the positive x-direction near P.
- If moving in the positive x-direction from P crosses over level curves corresponding to larger values of f, then $f_x(P) > 0$.
- Inversely, if moving in the positive x-direction crosses over level curves with smaller values of f, then $f_x(P) < 0$.

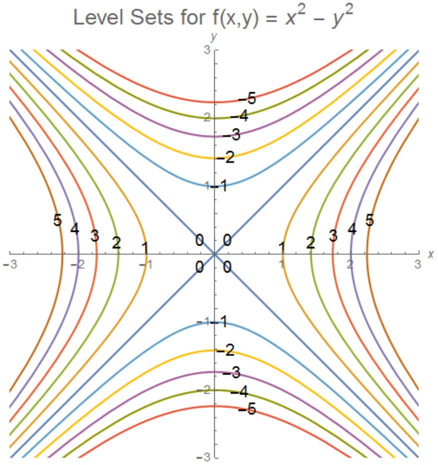
We can also estimate the approximate value of $f_x(P)$ based on how much f changes as we move in the x-direction:

- Per the limit definition, if the value of f changes by a total amount Δf as we move a distance Δx in the x-direction from P, then $f_x(P) \approx \Delta f / \Delta x$.

All of the same logic also applies with y in place of x.

Geometry of Partial Derivatives, V

For example, here are the level sets for $f(x, y) = x^2 - y^2$:



Consider $P = (1, 1)$, located on the level set where $f = 0$.

- If we move in the positive x-direction, we move towards the level sets where $f = 1, 2, 3, \ldots$. This means $f_x > 0$.
- If we move in the positive y-direction, we move towards the level sets where $f = -1, -2, -3, \ldots$. This means $f_y < 0$.

Higher Derivatives, I

Like in the one-variable case, we also have higher-order partial derivatives, obtained by taking a partial derivative of a partial derivative.

- However, because we have more than one choice of derivative at each stage, we get a number of possible second derivatives.
- For a function of two variables, there are four second-order partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[f_x] \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}[f_x]$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}[f_y] \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[f_y].$$

- Remark: Partial derivatives in subscript notation are applied left-to-right, while partial derivatives in differential operator notation are applied right-to-left. (In practice, the order of the partial derivatives rarely matters, as we will see.)

Higher Derivatives, II

Example: Find the second-order partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} for $f(x, y) = x^3y^4 + y\,e^{2x}$.

Higher Derivatives, II

<u>Example</u>: Find the second-order partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} for $f(x, y) = x^3y^4 + y\,e^{2x}$.

- First, we have $f_x = 3x^2y^4 + 2y\,e^{2x}$ and $f_y = 4x^3y^3 + e^{2x}$.
- So, $f_{xx} = \dfrac{\partial}{\partial x}\left[3x^2y^4 + 2y\,e^{2x}\right] = 6xy^4 + 4y\,e^{2x}$.
- Next, $f_{xy} = \dfrac{\partial}{\partial y}\left[3x^2y^4 + 2y\,e^{2x}\right] = 12x^2y^3 + 2e^{2x}$.
- Also, $f_{yx} = \dfrac{\partial}{\partial x}\left[4x^3y^3 + e^{2x}\right] = 12x^2y^3 + 2e^{2x}$.
- Finally, $f_{yy} = \dfrac{\partial}{\partial y}\left[4x^3y^3 + e^{2x}\right] = 12x^3y^2$.

Higher Derivatives, III

Example: Find the second-order partial derivatives f_{xz}, f_{yz}, f_{zx}, and f_{zy} for $f(x, y, z) = x^4 y^2 z^3$.

Higher Derivatives, III

Example: Find the second-order partial derivatives f_{xz}, f_{yz}, f_{zx}, and f_{zy} for $f(x, y, z) = x^4y^2z^3$.

- First, we have $f_x = 4x^3y^2z^3$, $f_y = x^4(2y)z^3 = 2x^4yz^3$, and $f_z = x^4y^2(3z^2) = 3x^4y^2z^2$.
- So, $f_{xz} = \dfrac{\partial}{\partial z}\left[4x^3y^2z^3\right] = 4x^3y^2(3z^2) = 12x^3y^2z^2$.
- Next, $f_{yz} = \dfrac{\partial}{\partial z}\left[2x^4yz^3\right] = 2x^4y(3z^2) = 6x^4yz^2$.
- Also, $f_{zx} = \dfrac{\partial}{\partial x}\left[3x^4y^2z^2\right] = 3(4x^3)y^2z^2 = 12x^3y^2z^2$.
- Finally, $f_{zy} = \dfrac{\partial}{\partial y}\left[3x^4y^2z^2\right] = 3x^4(2y)z^2 = 6x^4yz^2$.

Higher Derivatives, IV

Notice that in both of the examples, we had an equality of the "mixed partial derivatives": $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$. This is a general fact:

Theorem (Clairaut's Theorem)

If both partial derivatives f_{xy} and f_{yx} are continuous, then they are equal. The same applies for any pair of mixed partial derivatives.

- In other words, the mixed partials are always equal (given mild assumptions about continuity), so for a function of two variables, there are really only three second-order partial derivatives: f_{xx}, f_{xy}, and f_{yx}.
- This theorem can be proven using the limit definition of derivative and the Mean Value Theorem, but the details are unenlightening, so I will skip them.

Higher Derivatives, V

We can continue on and take higher-order partial derivatives.

- For example, a function $f(x, y)$ has eight third-order partial derivatives: f_{xxx}, f_{xxy}, f_{xyx}, f_{xyy}, f_{yxx}, f_{yxy}, f_{yyx}, and f_{yyy}.
- By Clairaut's Theorem, we can reorder the partial derivatives any way we want (if they are continuous, which is almost always the case).
- Thus, $f_{xxy} = f_{xyx} = f_{yxx}$ and $f_{xyy} = f_{yxy} = f_{yyx}$.
- So in fact, $f(x, y)$ only has four different third-order partial derivatives: f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}.
- Likewise, $f(x, y)$ only has five different fourth-order partial derivatives: f_{xxxx}, f_{xxxy}, f_{xxyy}, f_{xyyy}, f_{yyyy}.

Higher Derivatives, VI

Example: Find the third-order partial derivatives f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy} for $f(x, y) = x^4y^2 + x^3e^y$.

Higher Derivatives, VI

Example: Find the third-order partial derivatives f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy} for $f(x,y) = x^4y^2 + x^3e^y$.

- First, we have $f_x = 4x^3y^2 + 3x^2e^y$ and $f_y = 2x^4y + x^3e^y$.
- Next, $f_{xx} = (f_x)_x = 12x^2y^2 + 6xe^y$,
  $f_{xy} = (f_x)_y = 8x^3y + 3x^2e^y$, and
  $f_{yy} = (f_y)_y = 2x^4 + x^3e^y$.
- Finally, $f_{xxx} = (f_{xx})_x = 24xy^2 + 6e^y$,
  $f_{xxy} = (f_{xx})_y = 24x^2y + 6xe^y$,
  $f_{xyy} = (f_{xy})_y = 8x^3 + 3x^2e^y$, and
  $f_{yyy} = (f_{yy})_y = x^3e^y$.

Higher Derivatives, VII

Example: If all 4th-order partial derivatives of $f(x, y, z)$ are continuous and $f_{xyz} = x^3 e^{xyz}$, what is f_{zyyx}?

Higher Derivatives, VII

Example: If all 4th-order partial derivatives of $f(x, y, z)$ are continuous and $f_{xyz} = x^3 e^{xyz}$, what is f_{zyyx}?

- By Clairaut's theorem, we can differentiate in any order, and so $f_{zyyx} = f_{xyzy} = (f_{xyz})_y$.
- Since $f_{xyz} = x^3 e^{xyz}$ we see that $(f_{xyz})_y = x^3 e^{xyz} \cdot xz$ by the chain rule.

Summary

We briefly discussed limits for functions of several variables.

We introduced partial derivatives as limits and discussed how to compute and interpret them.

We discussed some properties of higher-order partial derivatives.

Next lecture: Directional derivatives and gradient vectors.