Math 2321 (Multivariable Calculus)

Lecture #5 of 38 $\sim$ January 28, 2021

Curves and Motion in 3-Space

- Velocity, Speed, and Acceleration
- Arclength
- Unit Tangent and Unit Normal Vectors

This material represents §1.3.2 from the course notes.

Motion in 3-Space, I

To finish up our discussion of vectors and 3D geometry, we discuss motion in 3-space and some topics about the geometry of curves.

- For all of today's discussion, we will be considering the motion of a particle along a parametric curve, where the variable t represents time.
- So, suppose we have a particle traveling through space, so that its position at time t is given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.
- As t varies, the position of the particle traces a parametric curve in space.

Motion in 3-Space, II

So, suppose that $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ represents the position of a particle.

- As discussed last class, the derivative $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ represents the velocity of the particle.
- The magnitude $||\mathbf{v}(t)||$ of the velocity then represents the speed of the particle.
- Also, the second derivative $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$ represents the acceleration of the particle.

Motion in 3-Space, III

Example: Find the velocity, speed, and acceleration of a particle traveling along the path $\mathbf{r}(t) = \left\langle t^3, 6t, 3t^2 \right\rangle$.

Motion in 3-Space, III

Example: Find the velocity, speed, and acceleration of a particle traveling along the path $\mathbf{r}(t) = \left\langle t^3, 6t, 3t^2 \right\rangle$.

- The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 3t^2, 6, 6t \right\rangle$.
- The speed is $||\mathbf{v}(t)|| = \sqrt{(3t^2)^2 + 6^2 + (6t)^2} = \sqrt{9t^4 + 36 + 36t^2} = 3t^2 + 6$.
- The acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 6t, 0, 6 \rangle$.

Motion in 3-Space, IV

Example: Find the velocity, speed, and acceleration of a particle traveling along the path $\mathbf{r}(t) = \left\langle 2 + e^t, 5 + t\sqrt{2}, 4 - e^{-t} \right\rangle$.

Motion in 3-Space, IV

Example: Find the velocity, speed, and acceleration of a particle traveling along the path $\mathbf{r}(t) = \left\langle 2 + e^t, 5 + t\sqrt{2}, 4 - e^{-t} \right\rangle$.

- The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle e^t, \sqrt{2}, e^{-t} \right\rangle$.
- The speed is $||\mathbf{v}(t)|| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (e^{-t})^2} = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$.
- The acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \langle e^t, 0, -e^{-t} \rangle$.

Motion in 3-Space, V

As we might expect, there is a relation between the speed of a particle and the total distance it travels.

- So suppose we want to find the distance travelled by the particle between time $t = a$ and time $t = b$.

Motion in 3-Space, V

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- So suppose we want to find the distance travelled by the particle between time $t = a$ and time $t = b$.
- If we imagine breaking up the particle's path into many small segments, where the speed and direction of travel are essentially constant, then by the elementary relation $[\text{distance}] = [\text{rate}] \cdot [\text{time}]$, the length of each segment will simply be the speed of the particle $||\mathbf{v}(t_i)||$ times the length Δt_i of the corresponding time interval.
- The total length of all the segments is $\sum ||\mathbf{v}(t_i)|| \Delta t_i$.
- This sum is a Riemann sum for the integral $\int_a^b ||\mathbf{v}(t)||\, dt$, so by the usual sort of Riemann-sum limit argument, this integral gives the length of the curve.

Motion in 3-Space, VI

To summarize, we have the following:

Definition

If a particle has position $\mathbf{r}(t)$ *at time* t*, then the* *<u>arclength</u>* *of the curve, representing the total distance travelled by the particle, between* $t = a$ *and* $t = b$ *is given by* $\int_a^b ||\mathbf{v}(t)||\, dt$*.*

The arclength is often written using the letter s: thus, we would write $s = \int_a^b ||\mathbf{v}(t)||\, dt$.

Motion in 3-Space, VII

Example: Find the arclength of the curve $\mathbf{r}(t) = \langle t^3, 6t, 3t^2 \rangle$ from $t = 1$ to $t = 2$.

Motion in 3-Space, VII

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- We calculated earlier that $||\mathbf{v}(t)|| = \sqrt{(3t^2)^2 + 6^2 + (6t)^2} = \sqrt{9t^4 + 36 + 36t^2} = 3t^2 + 6$.
- Thus, $s = \int_1^2 (3t^2 + 6)\, dt = (t^3 + 6t)\big|_{t=1}^2 = 13$.

Motion in 3-Space, VII

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Example: At time t seconds, a particle has position $\mathbf{r}(t) = \left\langle 2 + e^t,\ 5 + t\sqrt{2},\ 4 - e^{-t} \right\rangle$ meters. Find the distance the particle travels between $t = 0$s and $t = 2$s.

Motion in 3-Space, VII

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Example: At time t seconds, a particle has position $\mathbf{r}(t) = \langle 2 + e^t, 5 + t\sqrt{2}, 4 - e^{-t} \rangle$ meters. Find the distance the particle travels between $t = 0$s and $t = 2$s.

- We calculated earlier that $||\mathbf{v}(t)|| = e^t + e^{-t}$.
- Thus, $s = \int_0^2 (e^t + e^{-t})\, dt = \left(e^t - e^{-t}\right)\Big|_{t=0}^{2} = (e^2 - e^{-2})$m.

Motion in 3-Space, VIII

Using the vector language makes it very straightforward to analyze projectile motion.

Example: A projectile is fired at time $t = 0\text{s}$ from initial position $\mathbf{r}(0) = \langle 4, 6, 0 \rangle$ m in a vacuum with initial velocity $\mathbf{v}(0) = \langle 1, -2, 45 \rangle \ \frac{\text{m}}{\text{s}}$. Assuming that the only force acting on the projectile is the downward acceleration due to gravity of $\mathbf{a}(t) = \langle 0, 0, -10 \rangle \ \frac{\text{m}}{\text{s}^2}$, find the following:

1. The position of the projectile at time t.
2. The projectile's maximum height.
3. The time the projectile hits the ground.
4. The speed of the projectile when it hits the ground.
5. The total distance the projectile travels along its path.

Motion in 3-Space, IX

Example: A projectile has initial position $\mathbf{r}(0) = \langle 4, 6, 0 \rangle$ m, initial velocity $\mathbf{v}(0) = \langle 1, -2, 45 \rangle \, \frac{\text{m}}{\text{s}}$, and constant acceleration $\mathbf{a}(t) = \langle 0, 0, -10 \rangle \, \frac{\text{m}}{\text{s}^2}$. Find

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Motion in 3-Space, IX

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1. The position of the projectile at time t.

- We just have to take the antiderivative of $\mathbf{a}(t)$ twice to find $\mathbf{r}(t)$ and plug in the initial conditions.
- Since the acceleration is $\mathbf{a}(t) = \langle 0, 0, -10 \rangle \frac{\text{m}}{\text{s}^2}$, integrating yields $\mathbf{v}(t) = \langle C_1,\ C_2,\ C_3 - 10t \rangle \frac{\text{m}}{\text{s}}$ for some C_1, C_2, C_3.
- Setting $t = 0$ yields $\mathbf{v}(0) = \langle C_1, C_2, C_3 \rangle$, so since $\mathbf{v}(0)$ is given as $\langle 1, -2, 45 \rangle \frac{\text{m}}{\text{s}}$, we see that $\mathbf{v}(t) = \langle 1, -2, 45 - 10t \rangle \frac{\text{m}}{\text{s}}$.

Motion in 3-Space, IX

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- Setting $t = 0$ yields $\mathbf{v}(0) = \langle C_1, C_2, C_3 \rangle$, so since $\mathbf{v}(0)$ is given as $\langle 1, -2, 45 \rangle\ \frac{\text{m}}{\text{s}}$, we see that $\mathbf{v}(t) = \langle 1, -2, 45 - 10t \rangle\ \frac{\text{m}}{\text{s}}$.
- Integrating again yields $\mathbf{r}(t) = \left\langle C_4 + t,\ C_5 - 2t,\ C_6 + 45t - 5t^2 \right\rangle$ m. Since the initial position is $\mathbf{r}(0) = \langle 4, 6, 0 \rangle$ m, we see that $\mathbf{r}(0) = \langle C_4, C_5, C_6 \rangle$, and therefore $\mathbf{r}(t) = \left\langle 4 + t,\ 6 - 2t,\ 45t - 5t^2 \right\rangle$ m.

Motion in 3-Space, X

Example: At time ts, a projectile has position $\mathbf{r}(t) = \left\langle 4+t,\, 6-2t,\, 45t-5t^2 \right\rangle$ m. Find

2. The projectile's maximum height.

Motion in 3-Space, X

<u>Example</u>: At time ts, a projectile has position $\mathbf{r}(t) = \langle 4+t, 6-2t, 45t-5t^2 \rangle$ m. Find

2. The projectile's maximum height.
   - The height of the projectile is the z-coordinate. Since $z'(t) = 45 - 10t$, the maximum height occurs when $t = 4.5$s, and the height is $z(4.5\text{s}) = 101.25$m.
3. The time the projectile hits the ground.

Motion in 3-Space, X

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3. The time the projectile hits the ground.
   - The height of the projectile is the z-coordinate, so it hits the ground when $45t - 5t^2 = 0$, so that $t = 0$ or $t = 9$.
   - Thus, the projectile hits the ground at time $t = 9$s.
4. The speed of the projectile when it hits the ground.

Motion in 3-Space, X

Example: At time ts, a projectile has position $\mathbf{r}(t) = \langle 4+t,\, 6-2t,\, 45t-5t^2 \rangle$ m. Find

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- The height of the projectile is the z-coordinate. Since $z'(t) = 45 - 10t$, the maximum height occurs when $t = 4.5$s, and the height is $z(4.5\text{s}) = 101.25$m.

3. The time the projectile hits the ground.

- The height of the projectile is the z-coordinate, so it hits the ground when $45t - 5t^2 = 0$, so that $t = 0$ or $t = 9$.
- Thus, the projectile hits the ground at time $t = 9$s.

4. The speed of the projectile when it hits the ground.

- The speed is $||\mathbf{v}(t)|| = \sqrt{(1)^2 + (-2)^2 + (45-10t)^2}$.
- At time $t = 9$s, the speed is $||\mathbf{v}(9\text{s})|| = \sqrt{2030}\,\frac{\text{m}}{\text{s}} \approx 45.056\,\frac{\text{m}}{\text{s}}$.

Motion in 3-Space, XI

Example: At time ts, a projectile has position $\mathbf{r}(t) = \langle 4+t,\, 6-2t,\, 45t-5t^2 \rangle$ m. Find

5. The total distance the projectile travels along its path.

Motion in 3-Space, XI

Example: At time ts, a projectile has position $\mathbf{r}(t) = \left\langle 4+t,\, 6-2t,\, 45t-5t^2 \right\rangle$ m. Find

5. The total distance the projectile travels along its path.

- The total distance travelled by the particle is the arclength of the path between $t = 0$s and $t = 9$s. Since $||\mathbf{v}(t)|| = \sqrt{1^2 + (-2)^2 + (45-10t)^2}\,\frac{\text{m}}{\text{s}}$, the arclength is given by $\int_0^9 \sqrt{1^2 + (-2)^2 + (45-10t)^2}\,dt$ meters.
- This integral is not so easy to evaluate, but it is possible to compute it using a series of substitutions ($u = 45 - 10t$ followed by $u = \sqrt{5}\tan(s)$).
- The result is $\int_0^9 \sqrt{1^2 + (-2)^2 + (45-10t)^2}\,dt = \frac{1}{2}\left(9\sqrt{2030} + \sinh^{-1}\left(9\sqrt{5}\right)\right) \approx 204.598$m.

Unit Tangent and Unit Normal Vectors, I

When working with curves in 3-space (especially in physics), it is often useful to use normalized tangent vectors, rather than just the velocity and acceleration vectors.

There are three standard "normalized tangent vectors", which we will discuss in order:

- The unit tangent vector $\mathbf{T}(t)$
- The unit normal vector $\mathbf{N}(t)$
- The unit binormal vector $\mathbf{B}(t)$.

Unit Tangent and Unit Normal Vectors, II

Definition

If $\mathbf{r}(t)$ *represents the position of a particle, the* *unit tangent vector* $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{||\mathbf{v}(t)||}$ *measures the direction in which the particle is moving.*

- By definition, the unit tangent vector satisfies $||\mathbf{T}(t)|| = 1$, and $\mathbf{T}(t)$ is in the same direction as $\mathbf{r}'(t)$.
- Thus $\mathbf{T}(t)$ is a unit vector, in the same direction as the velocity (or "tangent") vector.

Unit Tangent and Unit Normal Vectors, III

Definition

If $\mathbf{r}(t)$ *represents the position of a particle, the* *unit normal vector* $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$*, orthogonal to the unit tangent vector, measures roughly the direction that forces are pulling the particle.*

- By the definition, the unit normal vector satisfies $||\mathbf{N}(t)|| = 1$, as long as $\mathbf{T}'(t) \neq 0$.
- It can be shown that the unit normal vector $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$.
- The idea is to take the derivative of the equation since $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$: using properties of the dot product yields the result $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, and then dividing by $||\mathbf{T}'(t)||$ shows that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.

Unit Tangent and Unit Normal Vectors, IV

The vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ span the particle's plane of motion. The third coordinate direction is perpendicular to those two:

Definition

If $\mathbf{r}(t)$ represents the position of a particle, the *unit binormal vector* $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \dfrac{\mathbf{v}(t) \times \mathbf{a}(t)}{||\mathbf{v}(t) \times \mathbf{a}(t)||}$, *orthogonal to the tangent and normal vectors, gives the direction perpendicular to the particle's plane of motion.*

- By its definition via the cross product, the binormal vector is orthogonal to both the (unit) normal and (unit) tangent vectors, and since those vectors have length 1, so does $\mathbf{B}$.
- Thus, we see that the unit tangent, unit normal, and unit binormal vectors are all orthogonal to each other.

Unit Tangent and Unit Normal Vectors, V

Example: Find the unit tangent and normal vectors for the curve $\mathbf{r}(t) = \langle \cos(2t), \sin(2t) \rangle$.

Unit Tangent and Unit Normal Vectors, V

Example: Find the unit tangent and normal vectors for the curve $\mathbf{r}(t) = \langle \cos(2t), \sin(2t) \rangle$.

- We have $\mathbf{v}(t) = \langle -2\sin(2t), 2\cos(2t) \rangle$, and so $||\mathbf{v}(t)|| = \sqrt{[-2\sin(2t)]^2 + [2\cos(2t)]^2} = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = 2$.
- Thus, $\mathbf{T}(t) = \langle -\sin(2t), \cos(2t) \rangle$.

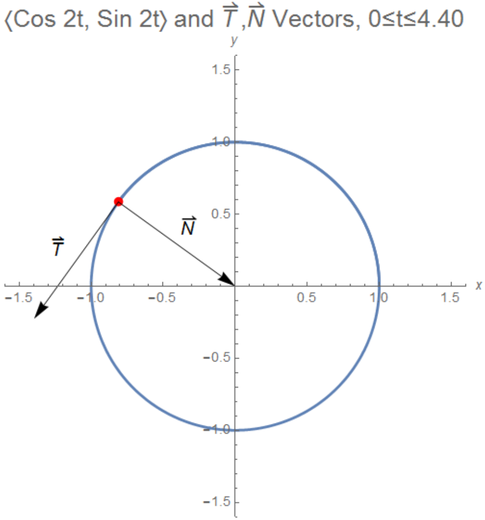
Unit Tangent and Unit Normal Vectors, V

Example: Find the unit tangent and normal vectors for the curve $\mathbf{r}(t) = \langle \cos(2t), \sin(2t) \rangle$.

- We have $\mathbf{v}(t) = \langle -2\sin(2t), 2\cos(2t) \rangle$, and so $||\mathbf{v}(t)|| = \sqrt{[-2\sin(2t)]^2 + [2\cos(2t)]^2} = \sqrt{4\sin^2(2t) + 4\cos^2(2t)} = 2$.
- Thus, $\mathbf{T}(t) = \langle -\sin(2t), \cos(2t) \rangle$.
- Then $\mathbf{T}'(t) = \langle -2\cos(2t), -2\sin(2t) \rangle$, and like above we can find $||\mathbf{T}'(t)|| = \sqrt{[-2\cos(2t)]^2 + [-2\sin(2t)]^2} = \sqrt{4\cos^2(2t) + 4\sin^2(2t)} = 2$.
- Therefore, $\mathbf{N}(t) = \langle -\cos(2t), -\sin(2t) \rangle$.

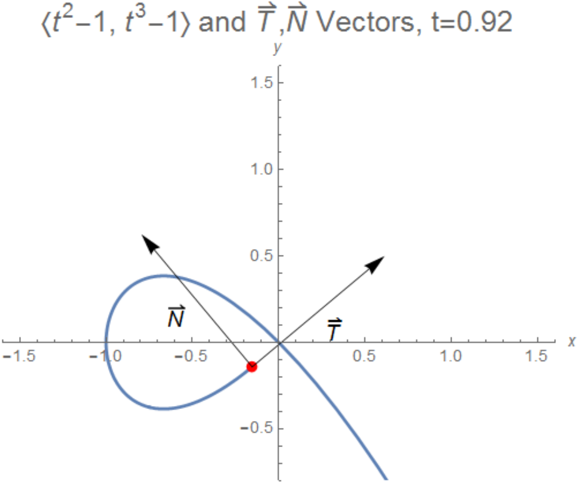
Unit Tangent and Unit Normal Vectors, VI

Here is a plot of the tangent and normal vectors in the previous example:



Unit Tangent and Unit Normal Vectors, VII

Here is a plot of the tangent and normal vectors for $\mathbf{r}(t) = \langle t^2 - 1, t^3 - t \rangle$:



Unit Tangent and Unit Normal Vectors, VIII

Example: For the helix $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 4t \rangle$, find **T**, **N**, **B**.

Unit Tangent and Unit Normal Vectors, VIII

Example: For the helix $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 4t \rangle$, find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$.

- We have $\mathbf{v}(t) = \langle 3\cos(3t), -3\sin(3t), 4 \rangle$, so that $||\mathbf{v}|| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 16} = 5$.
- Thus, $\mathbf{T} = \left\langle \frac{3\cos(3t)}{5}, -\frac{3\sin(3t)}{5}, \frac{4}{5} \right\rangle$.

Unit Tangent and Unit Normal Vectors, VIII

Example: For the helix $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 4t \rangle$, find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$.

- We have $\mathbf{v}(t) = \langle 3\cos(3t), -3\sin(3t), 4 \rangle$, so that $||\mathbf{v}|| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 16} = 5$.
- Thus, $\mathbf{T} = \left\langle \frac{3\cos(3t)}{5}, -\frac{3\sin(3t)}{5}, \frac{4}{5} \right\rangle$.
- Next, $\frac{d\mathbf{T}}{dt} = \left\langle -\frac{9\sin(3t)}{5}, -\frac{9\cos(3t)}{5}, 0 \right\rangle$ so that $\left|\left| \frac{d\mathbf{T}}{dt} \right|\right| = \frac{9}{5}$.
- Then $\mathbf{N} = \langle -\sin(3t), -\cos(3t), 0 \rangle$.

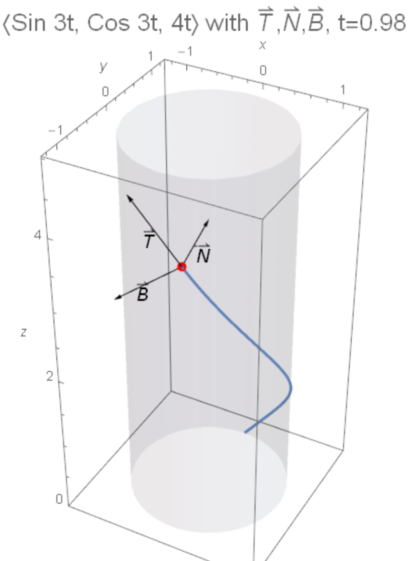
Unit Tangent and Unit Normal Vectors, VIII

Example: For the helix $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 4t \rangle$, find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$.

- We have $\mathbf{v}(t) = \langle 3\cos(3t), -3\sin(3t), 4 \rangle$, so that $||\mathbf{v}|| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 16} = 5$.
- Thus, $\mathbf{T} = \left\langle \frac{3\cos(3t)}{5}, -\frac{3\sin(3t)}{5}, \frac{4}{5} \right\rangle$.
- Next, $\frac{d\mathbf{T}}{dt} = \left\langle -\frac{9\sin(3t)}{5}, -\frac{9\cos(3t)}{5}, 0 \right\rangle$ so that $\left|\left|\frac{d\mathbf{T}}{dt}\right|\right| = \frac{9}{5}$.
- Then $\mathbf{N} = \langle -\sin(3t), -\cos(3t), 0 \rangle$.
- Finally, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3\cos(3t)}{5} & -\frac{3\sin(3t)}{5} & \frac{4}{5} \\ -\sin(3t) & -\cos(3t) & 0 \end{vmatrix}$

$$= \left\langle \frac{4\cos(3t)}{5}, -\frac{4\sin(3t)}{5}, -\frac{3}{5} \right\rangle.$$

Unit Tangent and Unit Normal Vectors, IX



Here is a plot of **T**, **N**, **B** for the helix example.

As the particle moves along the curve, the **T**, **N**, **B** vectors will travel along with the particle, giving a "local coordinate system" that behaves similarly to the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Other Quantities, I [FOR FUN ONLY]

There are a few other vector and scalar quantities that are occasionally useful when working with curves in 3-space in physical applications.

We will not make use of these in our course, but they show up in more advanced geometry, and are also very useful when doing astrophysical calculations (they are often used to give derivations of Kepler's laws for planetary motion).

Other Quantities, II [FOR FUN ONLY]

Definition

If $\mathbf{r}(t)$ represents the position of a particle, the *<u>curvature</u>* $\kappa(t) = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{v}(t)||} = \dfrac{||\mathbf{v}(t) \times \mathbf{a}(t)||}{||\mathbf{v}(t)||^3}$ *measures how much the path of the particle is curving.*

<u>Example</u>: Find the curvature of the circle $\mathbf{r}(t) = \langle r\cos t,\ r\sin t,\ 0\rangle$.

- We have $\mathbf{v}(t) = \langle -r\sin t,\ r\cos t,\ 0\rangle$ so $||\mathbf{v}(t)|| = r$ and thus $\mathbf{T}(t) = \langle -\sin t,\ \cos t,\ 0\rangle$.
- We then have $\mathbf{T}'(t) = \langle -\cos t,\ -\sin t,\ 0\rangle$, so $||\mathbf{T}'(t)|| = 1$, and therefore $\kappa(t) = 1/r$.
- More generally, for any circle of radius r, the curvature is $1/r$.

Curvature is always nonnegative. A small curvature indicates that the path is close to a line segment, while a large curvature means the path is turning sharply.

Other Quantities, III [FOR FUN ONLY]

The acceleration can be decomposed into a component in the direction of the unit tangent vector $\mathbf{T}$ and a component in the direction of the unit normal vector $\mathbf{N}$:

Definition

The *tangential* *and* *normal components of acceleration* *are* $a_T = \frac{d}{dt}[||\mathbf{v}||]$ *and* $a_N = \kappa\,||\mathbf{v}||^2$, *and they give the decomposition* $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

Example: For $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 4t \rangle$, find κ, a_T, and a_N.

- We previously found $||\mathbf{v}|| = 5$, so $a_T = \frac{d}{dt}[5] = 0$.
- We also calculated $\mathbf{T} = \langle \frac{3\cos(3t)}{5}, -\frac{3\sin(3t)}{5}, \frac{4}{5} \rangle$ and $\mathbf{N} = \langle -\sin(3t), -\cos(3t), 0 \rangle$.
- Then $\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{v}(t)||} = \frac{9/5}{5} = \frac{9}{25}$, so $a_N = \kappa||\mathbf{v}||^2 = 9$.
- Indeed, $\mathbf{a}(t) = \langle -9\sin(3t), -9\cos(3t), 0 \rangle = 0\mathbf{T} + 9\mathbf{N}$.

Other Quantities, IV [FOR FUN ONLY]

The final quantity we will mention is called the torsion:

Definition

If $\mathbf{r}(t)$ *represents a particle's position, the* *torsion*

$$\tau = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{||\mathbf{v}(t)||} = \frac{1}{||\mathbf{v} \times \mathbf{a}||^2} \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$

measures how sharply the path of the particle is twisting out of its plane of motion.

The torsion is the least intuitive of all of the quantities we have listed. (Note that it is the only one to involve the third derivatives of the coordinate functions.)

Other Quantities, V [FOR FUN ONLY]

To close out our discussion in this chapter, we remark that there are many relationships between the various quantities $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, τ, a_T, a_N we have described.

Three of these are the so-called "Frenet-Serret formulas", which describe how to compute the derivatives of the vectors **T**, **N**, and **B** with respect to arclength:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}. \qquad \frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}. \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Our primary emphasis in this course will be on understanding the vectors **T** and **N**. (We won't see them again for a while, but I promise: they will return at the end of the semester!)

Summary

We discussed how to solve various problems involving distances, lines, and planes.

We discussed vector-valued functions and curves in 3-space.

Next lecture: Limits and partial derivatives.