Math 2321 (Multivariable Calculus)

Lecture #4 of 38 $\sim$ January 27, 2021

Lines, Planes, and Vector-Valued Functions

- Distances, Lines, and Planes
- Vector-Valued Functions
- Parametric Curves in 3-Space

This material represents §1.2.4-1.3.1 from the course notes.

Recap, I

Some reminders:

Definition

The <u>dot product</u> of two vectors $\mathbf{v}_1 = \langle a_1, \ldots, a_n \rangle$ *and* $\mathbf{v}_2 = \langle b_1, \ldots, b_n \rangle$ *is defined to be the scalar* $\mathbf{v}_1 \cdot \mathbf{v}_2 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Definition

The <u>cross product</u> of $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ *and* $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$ *is defined to be the vector*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *are the standard unit vectors.*

Recap, II

Proposition (Parametrization of a Line)

Given distinct points $P_1 = \langle x_1, y_1, z_1 \rangle$ and $P_2 = \langle x_2, y_2, z_2 \rangle$, the points $\langle x, y, z \rangle$ on the line l through P_1 and P_2 are given parametrically by $\langle x, y, z \rangle = P_1 + t\,\mathbf{v}$, as t varies through the real numbers and $\mathbf{v} = P_2 - P_1 = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$. The equation can be written explicitly as
$\langle x, y, z \rangle = \langle x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1),\ z_1 + t(z_2 - z_1) \rangle$.

Proposition (Planes and Normal Vectors)

The plane defined by $ax + by + cz = d$ is orthogonal to its normal vector $\mathbf{n} = \langle a, b, c \rangle$. In other words, every line lying in this plane is orthogonal to $\langle a, b, c \rangle$. Conversely, given a nonzero vector $\mathbf{n} = \langle a, b, c \rangle$, there is a unique plane normal to that vector passing through a given point (x_0, y_0, z_0), and its equation is
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

More Lines and Planes, I

Example: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.

More Lines and Planes, I

Example: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.

- The normal vector to each plane will be orthogonal to the line of intersection (since the line lies in both planes).
- Therefore, we can get the direction vector of the line by taking the cross product of the two planes' normal vectors.

More Lines and Planes, I

<u>Example</u>: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.

- The normal vector to each plane will be orthogonal to the line of intersection (since the line lies in both planes).
- Therefore, we can get the direction vector of the line by taking the cross product of the two planes' normal vectors.
- We have $\mathbf{n}_1 = \langle 1, -1, 2 \rangle$ and $\mathbf{n}_2 = \langle 2, 1, -1 \rangle$. Then the cross product is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -1, 5, 3 \rangle$.
- Sanity check: $\mathbf{v} \cdot \mathbf{n}_1 = -1(1) + 5(-1) + 3(2) = 0$ and $\mathbf{v} \cdot \mathbf{n}_2 = -1(2) + 5(1) + 3(-1) = 0$.

More Lines and Planes, II

Example: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.

More Lines and Planes, II

Example: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.

- We have the direction vector $\mathbf{v} = \langle -1, 5, 3 \rangle$.
- Now we need to find a point in both planes (since we need a point on the line).
- There are many possible choices, so what we can do is try looking for one with $x = 0$: this requires $-y + 2z = 3$ and $y - z = 0$.
- Solving these two equations together yields $y = z = 3$, so $(0, 3, 3)$ is in both planes and thus on the line l.
- Applying the line parametrization formula gives $l : \langle x, y, z \rangle = \langle -t,\ 5t + 3,\ 3t + 3 \rangle$.

More Lines and Planes, III

Another common problem is to compute the distance from a point to a plane. It turns out there is a reasonably nice formula for this distance:

Proposition (Point-to-Plane Distance)

The distance from the point $P = (x_0, y_0, z_0)$ to the plane $ax + by + cz = d$ is equal to $\dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

The shortest vector connecting P to the plane will be in the same direction as the normal vector to the plane.

- In principle, we could use this information to parametrize the line joining P to the plane, find the intersection of the line and plane, and finally compute the distance we seek.
- However, there is a less messy way using vector projections.

More Lines and Planes, IV

Proof:

- Let Q be any point on the plane $ax + by + cz = d$ and let $\mathbf{n}$ be the normal vector to the plane.
- Then the vector $\mathbf{v}$ connecting P to the plane is given by the vector projection of $\mathbf{w} = Q - P$ onto the normal $\mathbf{n}$.
- Explicitly, $\mathbf{v} = \text{Proj}_{\mathbf{n}}(\mathbf{w}) = \left(\dfrac{\mathbf{n} \cdot \mathbf{w}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$. Then the length of $\mathbf{v}$ is $||\mathbf{v}|| = \dfrac{|\mathbf{n} \cdot \mathbf{w}|}{||\mathbf{n}||^2} ||\mathbf{n}|| = \dfrac{|\mathbf{n} \cdot \mathbf{w}|}{||\mathbf{n}||}$.
- From our earlier results, we can take $\mathbf{n} = \langle a, b, c \rangle$: then $\mathbf{n} \cdot \mathbf{w} = \mathbf{n} \cdot \mathbf{P} - \mathbf{n} \cdot \mathbf{Q} = (ax_0 + by_0 + cz_0) - (d)$, and $||\mathbf{n}|| = \sqrt{a^2 + b^2 + c^2}$. (The fact that $\mathbf{n} \cdot \mathbf{Q} = d$ is just a restatement of the fact that $\mathbf{Q}$ lies in the plane $ax + by + cz = d$.)
- Then $||\mathbf{v}|| = \dfrac{|\mathbf{n} \cdot \mathbf{w}|}{||\mathbf{n}||} = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$, as claimed.

More Lines and Planes, V

Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $z = 0$.

More Lines and Planes, V

Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $z = 0$.

- The plane can be written as $0x + 0y + 1z = 0$.
- Then the formula gives $\dfrac{|0 \cdot 1 + 0 \cdot 2 + 1 \cdot 4 - 0|}{\sqrt{0^2 + 0^2 + 1^2}} = \dfrac{|4|}{1} = 4$.
- This result should agree with your intuition: the plane $z = 0$ is the horizontal xy-plane, and so the distance of any point to this plane is simply the absolute value of its z-coordinate.

Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $x + 2y - 2z = 3$.

More Lines and Planes, V

Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $z = 0$.

- The plane can be written as $0x + 0y + 1z = 0$.
- Then the formula gives $\dfrac{|0 \cdot 1 + 0 \cdot 2 + 1 \cdot 4 - 0|}{\sqrt{0^2 + 0^2 + 1^2}} = \dfrac{|4|}{1} = 4$.
- This result should agree with your intuition: the plane $z = 0$ is the horizontal xy-plane, and so the distance of any point to this plane is simply the absolute value of its z-coordinate.

Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $x + 2y - 2z = 3$.

- The formula gives $\dfrac{|1 \cdot 1 + 2 \cdot 2 - 2 \cdot 4 - 3|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \dfrac{|-6|}{\sqrt{9}} = 2$.

More Lines and Planes, VI

Example: Find the distance between the planes $2x + y + 2z = 1$ and $2x + y + 2z = 7$.

More Lines and Planes, VI

Example: Find the distance between the planes $2x + y + 2z = 1$ and $2x + y + 2z = 7$.

- These two planes are parallel since they have the same normal vector $\langle 2, 1, 2 \rangle$.
- To compute the distance between them, we can pick any point on one plane and compute its distance to the other plane.

More Lines and Planes, VI

<u>Example</u>: Find the distance between the planes $2x + y + 2z = 1$ and $2x + y + 2z = 7$.

- These two planes are parallel since they have the same normal vector $\langle 2, 1, 2 \rangle$.
- To compute the distance between them, we can pick any point on one plane and compute its distance to the other plane.
- Since the point $(0, 1, 0)$ lies on $2x + y + 2z = 1$, the formula gives the distance as $\dfrac{|2 \cdot 0 + 1 \cdot 1 + 2 \cdot 0 - 7|}{\sqrt{2^2 + 1^2 + 2^2}} = \dfrac{|-6|}{\sqrt{9}} = \boxed{2}$.

By the same argument, the distance between the planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is $\dfrac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$.

Vector-Valued Functions, I

Our goal now is to extend our results about lines and planes to discuss more general curves in 3-space.

- We introduce the notion of a vector-valued function and of its graph, which will be a curve or surface in space.
- Then we analyze the geometry of curves in 3-space from a physical perspective.
- Our motivation here is to develop some ideas (namely, the relationship between derivatives and tangent vectors, and also of the unit tangent and unit normal vectors to a curve) that will be important later in the semester.

Vector-Valued Functions, II

The first task is to introduce functions involving vectors:

Definition

A vector-valued function of one variable $\mathbf{r}(t)$ *is a function whose output is a vector, each of whose components is a function of the parameter* t.

Examples:

- $\mathbf{r}_1(t) = \left\langle t^2, 2t \right\rangle$.
- $\mathbf{r}_2(t) = \langle t, t, t \rangle$.
- $\mathbf{r}_3(t) = \left\langle \cos(t^2),\ e^{2t},\ \tan^{-1}(\sqrt{t^2+1}) \right\rangle$.

We have already encountered a few vector-valued functions: the parametrization of a line, such as $\mathbf{r}(t) = \langle 2+3t, 1-t, 1+2t \rangle$ is an example of a vector-valued function.

Vector-Valued Functions, III

We can add and scalar-multiply vector-valued functions in the same manner as normal vectors.

Example: Suppose that $\mathbf{r}_1(t) = \left\langle e^t, \cos(t), t^2 - 1 \right\rangle$ and $\mathbf{r}_2(t) = \left\langle t, 0, -t^2 \right\rangle$.

- Then $\mathbf{r}_1(t) + \mathbf{r}_2(t) = \langle e^t + t, \cos(t), -1 \rangle$.
- Also, $2\mathbf{r}_2(t) = \left\langle 2t, 0, -2t^2 \right\rangle$.

Vector-Valued Functions, IV

For the moment, we will primarily be interested in vector functions of the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, which have a single input parameter t and output a vector with 2 or 3 coordinates.

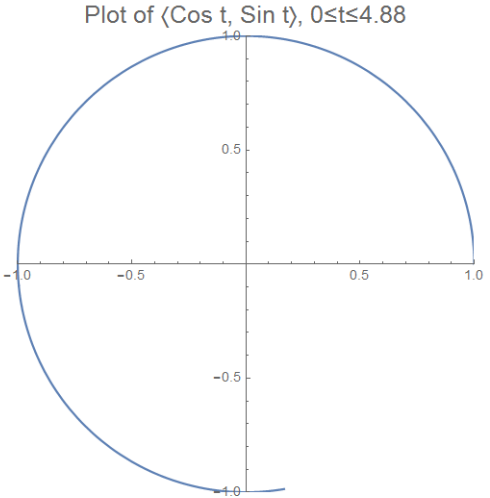
- As we vary t, these functions will trace out parametric curves in 2 or 3-dimensional space (respectively).

Later, we will also be interested in vector functions of the form $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$, for two input parameters s, t.

- These functions, in general, will describe surfaces in 3-dimensional space as we vary s and t.
- However, we need to develop more tools to be able to describe and analyze parametric surfaces, so we will focus on the one-variable case for now, and return to discuss surfaces in November.

Vector-Valued Functions, V

Example: The curve given by $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ traces around the unit circle as t varies.



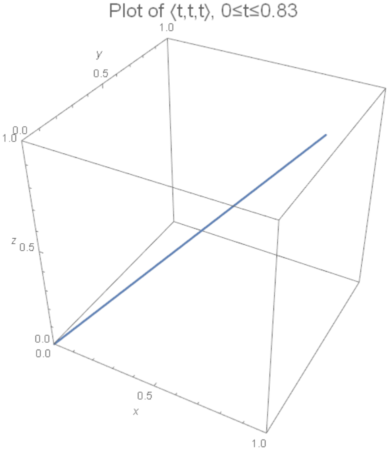
Vector-Valued Functions, VI

If we graph a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ as t varies, we will obtain a curve in 3-space.

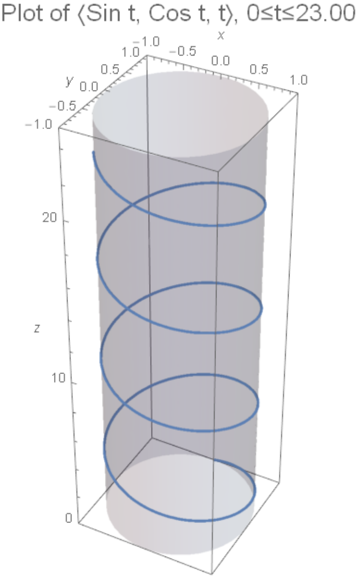
- A parametric curve in 3-space is just the set of points $(x(t),\ y(t),\ z(t))$ for some functions $x(t)$, $y(t)$, and $z(t)$.
- Remember that we are thinking of functions giving parametric curves interchangeably with vector functions: instead of getting a point $(x(t),\ y(t),\ z(t))$ as output from a function, we can equally well think of getting the vector output $\langle x(t),\ y(t),\ z(t) \rangle$.

Vector-Valued Functions, VII

Example: The curve given by $\mathbf{r}(t) = \langle t, t, t \rangle$ is a line passing through the origin with direction vector $\langle 1, 1, 1 \rangle$.



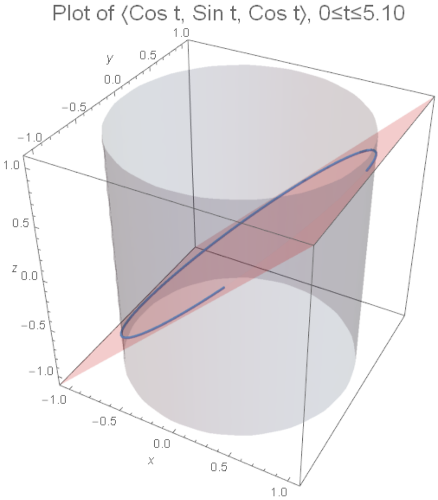
Vector-Valued Functions, VIII



Example: The curve given by $\mathbf{r}(t) = (\sin(t), \cos(t),\ t)$ is a helix wrapping around the cylinder $x^2 + y^2 = 1$.

As t increases, x and y trace around the unit circle at constant speed, while z increases at constant speed.

Vector-Valued Functions, IX



Example: The curve given by $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(t) \rangle$ is an ellipse.

We can see that this curve is an ellipse by observing that it is the intersection of the plane $z = x$ with the cylinder $x^2 + y^2 = 1$.

Vector-Valued Functions, X

Finding a parametrization of a particular curve with a given description can be difficult, and there is no general recipe.

- However, in some cases, we can find parametrizations of particular curves that are intersections of surfaces.
- One idea that can be effective is to set one variable equal to the parameter t, and then use the surface equations to solve for the other variables in terms of t.
- In some cases it can be better to use a slightly different choice for one variable, such as $\sin(t)$, $\cos(t)$, or e^t.

Vector-Valued Functions, XI

Example: Find a parametrization for the curve of intersection between the surfaces $z = x^2 + y^2$ and $y = x^2$.

Vector-Valued Functions, XI

Example: Find a parametrization for the curve of intersection between the surfaces $z = x^2 + y^2$ and $y = x^2$.

- In this case, if we simply try taking $x = t$, then the second equation requires $y = t^2$.
- Once we know $x = t$ and $y = t^2$, the first equation then tells us that $z = t^2 + t^4$, so we get the parametrization $\mathbf{r}(t) = \langle t, t^2, t^2 + t^4 \rangle$.
- Each point of this form lies on both of the surfaces.

Vector-Valued Functions, XII

Example: Find a parametrization for the curve of intersection between the surfaces $x^2 + z^2 = 4$ and $x + y + z = 3$.

Vector-Valued Functions, XII

Example: Find a parametrization for the curve of intersection between the surfaces $x^2 + z^2 = 4$ and $x + y + z = 3$.

- The first equation involves only x and z.
- From what we know about circles, we can describe all of the points (x, z) satisfying the first equation by taking $x = 2\cos(t)$, $z = 2\sin(t)$.
- Then we can solve for y by plugging these expressions into the second equation: this yields $y = 3 - x - z = 3 - 2\cos(t) - 2\sin(t)$.
- Putting this together gives the parametrization $\mathbf{r}(t) = \langle 2\cos(t), 3 - 2\cos(t) - 2\sin(t), 2\sin(t) \rangle$.

Derivatives of Vector-Valued Functions, I

The next natural question to ask is: can we take derivatives of vector-valued functions? The answer is, of course, yes!

Definition

The derivative of the vector function $\mathbf{r}(t)$ *is given by* $\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$, *provided the limit exists.*

- Note the extreme similarity of this definition with the definition of the derivative of a (scalar) function of one variable $f(t)$, which reads $f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$.
- For $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then by applying the definition of the derivative, we see that $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- Thus, quite sensibly, taking the derivative of a vector function is the same thing as differentiating each component.

Derivatives of Vector-Valued Functions, II

Example: For $\mathbf{r}(t) = \left\langle e^t, \cos(3t), t^4 - t^3 \right\rangle$, find the first derivative $\mathbf{r}'(t)$ and the second derivative $\mathbf{r}''(t)$.

Derivatives of Vector-Valued Functions, II

Example: For $\mathbf{r}(t) = \left\langle e^t, \cos(3t), t^4 - t^3 \right\rangle$, find the first derivative $\mathbf{r}'(t)$ and the second derivative $\mathbf{r}''(t)$.

- We just take the derivative of each component function to get $\mathbf{r}'(t)$, and then do it again to get $\mathbf{r}''(t)$.
- So, we have $\mathbf{r}'(t) = \left\langle e^t, -3\sin(3t), 4t^3 - 3t^2 \right\rangle$.
- Then also $\mathbf{r}''(t) = \left\langle e^t, -9\cos(3t), 12t^2 - 6t \right\rangle$.

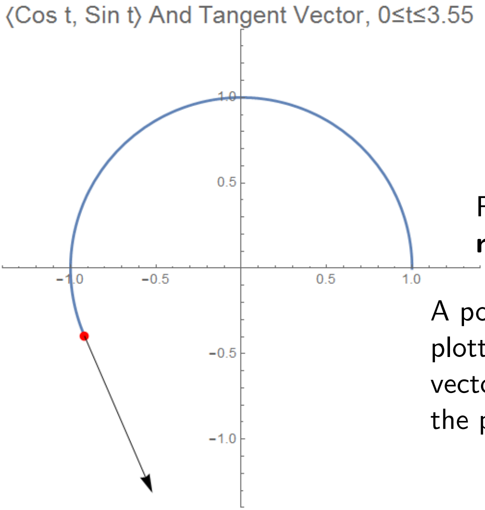
Derivatives of Vector-Valued Functions, III

The derivative $\mathbf{r}'(t)$ of a vector-valued function $\mathbf{r}(t)$ yields the tangent vector to the curve.

- To see this, imagine a particle traveling along the curve: at time t, its x-coordinate is $x(t)$ and its y-coordinate $y(t)$ at time t.
- Then $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ represents the instantaneous velocity vector of the particle, simply because it is the rate of change of the position vector.
- But it is not hard to see that if we actually draw secant vectors to the curve, they will approach the tangent vector to the curve.

This is much easier to see with a picture.

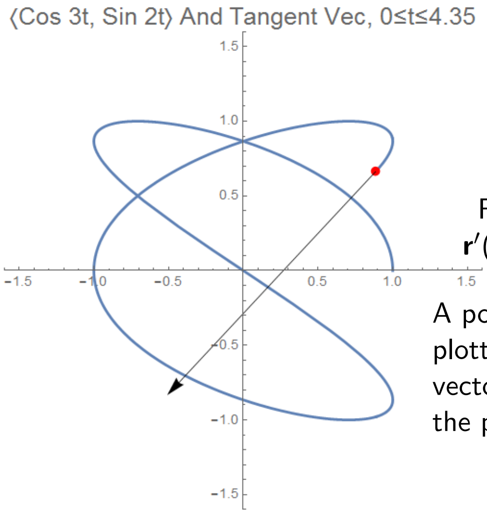
Derivatives of Vector-Valued Functions, IV



<u>Example</u>:
For $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$,
$\mathbf{r}'(t) = \langle -\sin(t), \cos(t) \rangle$.

A portion of the curve $\mathbf{r}(t)$ is plotted, along with the tangent vector $\mathbf{r}'(t)$ at the endpoint of the plotted portion of the curve.

Derivatives of Vector-Valued Functions, V



<u>Example</u>:

For $\mathbf{r}(t) = \langle \cos(3t), \sin(2t) \rangle$,
$\mathbf{r}'(t) = \langle -3\sin(3t), 2\cos(2t) \rangle$.

A portion of the curve $\mathbf{r}(t)$ is plotted, along with the tangent vector $\mathbf{r}'(t)$ at the endpoint of the plotted portion of the curve.

Derivatives of Vector-Valued Functions, VI

We can use the fact that $\mathbf{r}'$ gives a tangent vector to find the parametrization of the tangent line to a curve.

- Remember that to parametrize any line, we require a point and a direction vector.
- For the tangent line, the point is the point of tangency, and the direction vector is the derivative vector.

Derivatives of Vector-Valued Functions, VII

Example: Find a parametrization for the tangent line to the curve $\mathbf{r}(t) = \left\langle t^3, t^4, t^5 \right\rangle$ at $t = 1$.

Derivatives of Vector-Valued Functions, VII

Example: Find a parametrization for the tangent line to the curve $\mathbf{r}(t) = \left\langle t^3, t^4, t^5 \right\rangle$ at $t = 1$.

- Note $\mathbf{r}'(t) = \left\langle 3t^2, 4t^3, 5t^4 \right\rangle$.
- At $t = 1$, the point of tangency is $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$, while the direction vector is $\mathbf{r}'(1) = \langle 3, 4, 5 \rangle$.
- Therefore, the tangent line has parametrization $\langle x, y, z \rangle = \langle 1, 1, 1 \rangle + s \langle 3, 4, 5 \rangle = \langle 1 + 3s,\ 1 + 4s,\ 1 + 5s \rangle$.
- We used the parameter s instead because t was already used to describe the curve.

Derivatives of Vector-Valued Functions, VIII

Example: Find a parametrization for the tangent line to the curve $\mathbf{r}(t) = \left\langle t, e^{2t} \right\rangle$ at $t = 2$.

Derivatives of Vector-Valued Functions, VIII

Example: Find a parametrization for the tangent line to the curve $\mathbf{r}(t) = \langle t, e^{2t} \rangle$ at $t = 2$.

- Note $\mathbf{r}'(t) = \langle 1, 2e^{2t} \rangle$.
- At $t = 2$, the point of tangency is $\mathbf{r}(2) = \langle 2, e^4 \rangle$, while the direction vector is $\mathbf{r}'(2) = \langle 1, 2e^4 \rangle$.
- Therefore, the tangent line has parametrization $\langle x, y \rangle = \langle 2, e^4 \rangle + s \langle 1, 2e^4 \rangle = \langle 2 + s, e^4 + 2se^4 \rangle$.

Derivatives of Vector-Valued Functions, VIII

Example: Find a parametrization for the tangent line to the curve $\mathbf{r}(t) = \langle t, e^{2t} \rangle$ at $t = 2$.

- Note $\mathbf{r}'(t) = \langle 1, 2e^{2t} \rangle$.
- At $t = 2$, the point of tangency is $\mathbf{r}(2) = \langle 2, e^4 \rangle$, while the direction vector is $\mathbf{r}'(2) = \langle 1, 2e^4 \rangle$.
- Therefore, the tangent line has parametrization $\langle x, y \rangle = \langle 2, e^4 \rangle + s \langle 1, 2e^4 \rangle = \langle 2 + s, e^4 + 2se^4 \rangle$.
- Note that this is really just asking for a parametrization of the tangent line to the graph of $y = e^{2x}$. (You can check, if you like, that this tangent line is the same as the usual one you'd find by computing y'.)

Derivatives of Vector-Valued Functions, IX

As a last remark, we will observe that derivatives of vector functions satisfy rules strongly reminiscent of the product rule with regard to the dot and cross products:

- Specifically, we have
  $\dfrac{d}{dt}\left[\mathbf{r}_1 \cdot \mathbf{r}_2\right] = \mathbf{r}_1 \cdot (\mathbf{r}_2') + (\mathbf{r}_1') \cdot \mathbf{r}_2$ and
  $\dfrac{d}{dt}\left[\mathbf{r}_1 \times \mathbf{r}_2\right] = \mathbf{r}_1 \times (\mathbf{r}_2') + (\mathbf{r}_1') \times \mathbf{r}_2$.
- These properties can be verified by expanding out the dot and cross products of $[\mathbf{r}_1(t+h) - \mathbf{r}_1(t)]$ with $[\mathbf{r}_2(t+h) - \mathbf{r}_2(t)]$, applying the limit definition of derivative, and simplifying.

Summary

We discussed how to solve various problems involving distances, lines, and planes.

We discussed vector-valued functions and curves in 3-space.

Next lecture: Curves and motion in 3-space.