# Math 2321 (Multivariable Calculus)

## Lecture #3 of 38 $\sim$ January 25, 2021

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Cross Products, Lines and Planes

- Cross Products
- Lines and Planes (part 1)

This material represents §1.2.3-1.2.4 from the course notes.

# Reminders

Some reminders from last week:

**Definition**

*The dot product of two vectors* $\mathbf{v}_1 = \langle a_1, \ldots, a_n \rangle$ *and* $\mathbf{v}_2 = \langle b_1, \ldots, b_n \rangle$ *is defined to be the scalar* $\mathbf{v}_1 \cdot \mathbf{v}_2 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

**Definition**

*We define the norm of the vector* $\mathbf{v} = \langle a_1, \ldots, a_n \rangle$ *as* $||\mathbf{v}|| = \sqrt{(a_1)^2 + \cdots + (a_n)^2}$. *Note that* $||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$.

**Theorem (Dot Product)**

*For vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *forming an angle* $\theta$ *between them, we have* $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1||\ ||\mathbf{v}_2|| \cos(\theta)$.

# Cross Products, I

In addition to the dot product, we have another type of product defined for vectors in 3-space, called the cross product:

**Definition**

*The cross product of* $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ *and* $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$ *is defined to be the vector*
$\mathbf{v}_1 \times \mathbf{v}_2 = \langle y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_2 \rangle$.

- Important Note: The cross product is only defined for vectors with 3 components, and outputs another vector with 3 components. In contrast, the dot product is defined for vectors of any length, and outputs a scalar.

# Cross Products, II

A way to remember the cross product formula is as a determinant:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the standard unit vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

(Note $2 \times 2$ determinants are evaluated as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.)

It may seem a little unusual to have vectors inside a determinant, but it works out to the correct answer.

<u>Warning</u>: Don't forget the minus sign on the middle term in the determinant formula above!

# Cross Products, III

The fundamental property of the cross product $\mathbf{v}_1 \times \mathbf{v}_2$ is that it is orthogonal both to $\mathbf{v}_1$ and to $\mathbf{v}_2$.

- To verify this, we can just evaluate the dot products $\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$ and $\mathbf{v}_2 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$ and check they are both zero.
- For example, we have $\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = x_1(y_1 z_2 - y_2 z_1) + y_1(z_1 x_2 - z_2 x_1) + z_1(x_1 y_2 - x_2 y_2)$, which some algebra will confirm is equal to zero. (Three terms each appear once with a plus and once with a minus.)

# Cross Products, IV

Example: If $\mathbf{v} = \langle 4, 2, 1 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, find $\mathbf{v} \times \mathbf{w}$ and verify that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

# Cross Products, IV

<u>Example</u>: If $\mathbf{v} = \langle 4, 2, 1\rangle$ and $\mathbf{w} = \langle -2, 3, 1\rangle$, find $\mathbf{v} \times \mathbf{w}$ and verify that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

- We have

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 1 \\ -2 & 3 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 1 \\ -2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 2 \\ -2 & 3 \end{vmatrix} \mathbf{k} \\
&= [2 \cdot 1 - 3 \cdot 1]\mathbf{i} - [4 \cdot 1 - (-2) \cdot 1]\mathbf{j} + [4 \cdot 3 - (-2) \cdot 2]\mathbf{k} \\
&= -1\mathbf{i} - 6\mathbf{j} + 16\mathbf{k} = \langle -1, -6, 16\rangle .
\end{aligned}$$

- To check the orthogonality, we have $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 4, 2, 1\rangle \cdot \langle -1, -6, 16\rangle = -4 - 12 + 16 = 0$, and also $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = \langle -2, 3, 1\rangle \cdot \langle -1, -6, 16\rangle = 2 - 18 + 16 = 0$.

# Cross Products, V

Example: If $\mathbf{v} = \langle 1, 1, 3 \rangle$ and $\mathbf{w} = \langle 2, -1, 1 \rangle$, find

1. $\mathbf{v} \times \mathbf{w}$.
2. $\mathbf{w} \times \mathbf{v}$.
3. $\mathbf{v} \times \mathbf{v}$.
4. $\mathbf{w} \times \mathbf{w}$.

# Cross Products, V

Example: If $\mathbf{v} = \langle 1, 1, 3 \rangle$ and $\mathbf{w} = \langle 2, -1, 1 \rangle$, find

1. $\mathbf{v} \times \mathbf{w}$.
2. $\mathbf{w} \times \mathbf{v}$.
3. $\mathbf{v} \times \mathbf{v}$.
4. $\mathbf{w} \times \mathbf{w}$.

- For the first two, we have

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\ &= (1 - (-3))\mathbf{i} - (1 - 6)\mathbf{j} + (-1 - 2)\mathbf{k} = \langle 4, 5, -3 \rangle . \end{aligned}$$

$$\begin{aligned} \mathbf{w} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (-3 - 1)\mathbf{i} - (6 - 1)\mathbf{j} + (2 - (-1))\mathbf{k} = \langle -4, -5, 3 \rangle . \end{aligned}$$

- Notice that $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$ are negatives of each other.

# Cross Products, VI

Example: If $\mathbf{v} = \langle 1, 1, 3 \rangle$ and $\mathbf{w} = \langle 2, -1, 1 \rangle$, find

1. $\mathbf{v} \times \mathbf{w}$.
2. $\mathbf{w} \times \mathbf{v}$.
3. $\mathbf{v} \times \mathbf{v}$.
4. $\mathbf{w} \times \mathbf{w}$.

# Cross Products, VI

<u>Example</u>: If $\mathbf{v} = \langle 1, 1, 3 \rangle$ and $\mathbf{w} = \langle 2, -1, 1 \rangle$, find

1. $\mathbf{v} \times \mathbf{w}$.
2. $\mathbf{w} \times \mathbf{v}$.
3. $\mathbf{v} \times \mathbf{v}$.
4. $\mathbf{w} \times \mathbf{w}$.

- For the last two, we have

$$\begin{aligned} \mathbf{v} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= (3-3)\mathbf{i} - (3-3)\mathbf{j} + (1-1)\mathbf{k} = \langle 0, 0, 0 \rangle . \end{aligned}$$

$$\begin{aligned} \mathbf{w} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\ &= (-1-(-1))\mathbf{i} - (2-2)\mathbf{j} + (-2-(-2))\mathbf{k} = \langle 0, 0, 0 \rangle . \end{aligned}$$

- Note that both $\mathbf{v} \times \mathbf{v}$ and $\mathbf{w} \times \mathbf{w}$ are zero.

## Cross Products, VII

Here are a few algebraic properties of the cross product:

**Proposition (Properties of Cross Products)**

*For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$, and any scalar $r$, the following hold:*

1. *The cross product of two vectors is orthogonal to both vectors:* $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ *and* $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.
2. *The cross product distributes over addition:* $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{w} = (\mathbf{v}_1 \times \mathbf{w}) + (\mathbf{v}_2 \times \mathbf{w})$.
3. *The cross product distributes through scaling:* $(r\mathbf{v}) \times \mathbf{w} = r(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (r\mathbf{w})$.
4. *The cross product is anticommutative:* $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$. *In particular,* $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ *for any* $\mathbf{v}$.

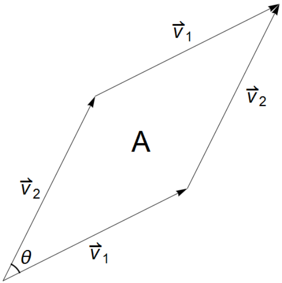
Proofs: Each of these properties is a simple algebraic calculation from the definition of the cross product.

# Cross Products, VIII

Like with the dot product, there is a relationship involving the cross product and the angle between two vectors:

**Theorem (Cross Product Theorem)**

*If $\theta$ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, then $||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin(\theta) = A$, where $A$ is the area of the parallelogram formed by $\mathbf{v}_1$ and $\mathbf{v}_2$.*



# Cross Products, IX

Proof:

- We just need to show that $||\mathbf{v}_1 \times \mathbf{v}_2||^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = ||\mathbf{v}_1||^2 \, ||\mathbf{v}_2||^2$, because we know that $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \cos(\theta)$ from the Dot Product Theorem.
- To check this, we simply write it out, which reduces to showing that $(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_2)^2 + (x_1 x_2 + y_1 y_2 + z_1 z_2)^2$ is equal to $\left[(x_1)^2 + (y_1)^2 + (z_1)^2\right] \cdot \left[(x_2)^2 + (y_2)^2 + (z_2)^2\right]$.
- Expanding both sides will show that they are equal.
- Now use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and the fact that the norm of the cross product is nonnegative to deduce that $||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin(\theta)$ as claimed.
- For the statement about the area, it is a standard fact of geometry that the area of the triangle with sides $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{1}{2} ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin(\theta)$. The parallelogram's area is twice this.

# Cross Products, X

<u>Example</u>: Find the area of the parallelogram formed by the vectors $\mathbf{v} = \langle 1, 4, 2 \rangle$ and $\mathbf{w} = \langle 0, 5, 6 \rangle$.

# Cross Products, X

<u>Example</u>: Find the area of the parallelogram formed by the vectors $\mathbf{v} = \langle 1, 4, 2 \rangle$ and $\mathbf{w} = \langle 0, 5, 6 \rangle$.

- By the Cross Product Theorem, the area of the parallelogram is $A = ||\mathbf{v} \times \mathbf{w}||$.
- We compute $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 0 & 5 & 6 \end{vmatrix} =$
$\begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} \mathbf{k} = \langle 14, -6, 5 \rangle$.
- This means the area of the parallelogram is $||\mathbf{v} \times \mathbf{w}|| = \sqrt{14^2 + (-6)^2 + 5^2} = \sqrt{257} \approx 16.031$.

# Cross Products, XI

Example: Find the area of the triangle whose vertices are the points $A(1, -1, 2)$, $B(2, -3, 1)$, and $C(2, 2, 2)$.

## Cross Products, XI

<u>Example</u>: Find the area of the triangle whose vertices are the points $A(1,-1,2)$, $B(2,-3,1)$, and $C(2,2,2)$.

- By the Cross Product Theorem, the area of the triangle is $A = \frac{1}{2}\,||\mathbf{v} \times \mathbf{w}||$, where $\mathbf{v}$ and $\mathbf{w}$ are vectors representing two sides of the triangle.
- We take $\mathbf{v} = B - A = \langle 1, -2, -1 \rangle$ and $\mathbf{w} = C - A = \langle 1, 3, 0 \rangle$.

# Cross Products, XI

<u>Example</u>: Find the area of the triangle whose vertices are the points $A(1,-1,2)$, $B(2,-3,1)$, and $C(2,2,2)$.

- By the Cross Product Theorem, the area of the triangle is $A = \frac{1}{2}\,||\mathbf{v}\times\mathbf{w}||$, where $\mathbf{v}$ and $\mathbf{w}$ are vectors representing two sides of the triangle.
- We take $\mathbf{v} = B - A = \langle 1,-2,-1\rangle$ and $\mathbf{w} = C - A = \langle 1,3,0\rangle$.
- Then $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ 1 & 3 & 0 \end{vmatrix} =$
$\begin{vmatrix} -2 & -1 \\ 3 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}\mathbf{k} = \langle 3,-1,5\rangle.$
- Hence the area of the triangle is $\frac{1}{2}\,||\langle 3,-1,5\rangle|| = \frac{1}{2}\sqrt{3^2+(-1)^2+5^2} = \sqrt{35}/2 \approx 2.958.$

# Lines in 3-Space, I

Among the many useful applications of vectors is that they can be used to give simple descriptions of lines and planes in 3-space.

- In the 2-dimensional plane, the general equation of a line is $ax + by = d$ for some constants $a, b, d$ (with $a$ and $b$ not both zero). The 3-dimensional version of this would be $ax + by + cz = d$ (with $a, b, c$ not all zero), which will describe a plane.
- To describe lines in 3-space, we instead need to describe them as parametric curves, in the form $x = x(t)$, $y = y(t)$, $z = z(t)$ for some functions $x(t)$, $y(t)$, $z(t)$. As $t$ varies, the set of points $(x(t), y(t), z(t))$ will form a curve: the goal is to choose the functions so that the curve is a line.
- It should not be surprising that the proper choice is to take $x(t), y(t), z(t)$ to be linear functions of $t$.

# Lines in 3-Space, II

**Proposition (Parametrization of a Line)**

*Given distinct points $P_1 = \langle x_1, y_1, z_1 \rangle$ and $P_2 = \langle x_2, y_2, z_2 \rangle$, the points $\langle x, y, z \rangle$ on the line $l$ through $P_1$ and $P_2$ are given parametrically by $\langle x, y, z \rangle = P_1 + t\,\mathbf{v}$, as $t$ varies through the real numbers and $\mathbf{v} = P_2 - P_1 = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$. The equation can be written explicitly as*
$\langle x, y, z \rangle = \langle x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1),\ z_1 + t(z_2 - z_1) \rangle$.

Proof:

- There is a unique line between two points, by the axioms of geometry, so we need only verify that $l : \langle x, y, z \rangle = P_1 + t\,\mathbf{v}$ is a line and that it goes through $P_1$ and $P_2$.
- The $x$,$y$, and $z$ coordinates are all linear, so $l$ is a line.
- Setting $t = 0$ yields $P_1$, while setting $t = 1$ yields $P_1 + (P_2 - P_1) = P_2$. Thus, $l$ passes through both $P_1$ and $P_2$.

# Lines in 3-Space, III

Some remarks:

- This procedure will yield the parametrization of a line in a space of any dimension, not just 3-space. Later, we will sometimes need to use it to parametrize lines in the xy-plane.
- We call the vector $\mathbf{v} = P_2 - P_1$ the direction vector for the line $l$: it tells us in which direction the line is pointing. The term $P_1$ in the sum $P_1 + t\,\mathbf{v}$ specifies which particular line we want, of all possible lines in that direction. The direction vector for a line is not unique: we could instead use any nonzero multiple of $\mathbf{v}$ and we would get the same line.

We also emphasize here that there are many different possible parametrizations of a given line (we could use a different starting point or a different multiple of the direction vector).

# Lines in 3-Space, IV

Example: Find a parametrization of the line through the point $(1, 0, 6)$ with direction vector $\mathbf{v} = \langle 4, 2, 3 \rangle$.

## Lines in 3-Space, IV

Example: Find a parametrization of the line through the point $(1, 0, 6)$ with direction vector $\mathbf{v} = \langle 4, 2, 3 \rangle$.

- The line is given parametrically by $\langle x, y, z \rangle = \langle 1, 0, 6 \rangle + t \langle 4, 2, 3 \rangle = \langle 1 + 4t,\ 2t,\ 6 + 3t \rangle$.
- More explicitly, we can convert the "vector equation" above to an explicit system of equations to see that this line is described by $x = 1 + 4t$, $y = 2t$, $z = 6 + 3t$.
- As $t$ ranges through the real numbers, the points $(x, y, z)$ will range along the desired line.
- For example, setting $t = 1$ yields the point $\langle 5, 2, 9 \rangle$ on the line, while $t = 2$ yields a different point $\langle 9, 4, 12 \rangle$.

# Lines in 3-Space, V

Example: Let $\ell$ be the line through $(1, 2, 3)$ and $(-1, 2, -1)$.

1. Find a parametrization of the line $\ell$.
2. Find the point on $\ell$ having $x$-coordinate 6.

# Lines in 3-Space, V

Example: Let $\ell$ be the line through $(1, 2, 3)$ and $(-1, 2, -1)$.

1. Find a parametrization of the line $\ell$.
2. Find the point on $\ell$ having $x$-coordinate 6.

1. First, we find the direction vector: we get $\mathbf{v} = \langle -1, 2, -1\rangle - \langle 1, 2, 3\rangle = \langle -2, 0, -4\rangle$.

- Then the line is given parametrically by $\langle x, y, z\rangle = \langle 1, 2, 3\rangle + t\,\langle -2, 0, -4\rangle = \langle 1-2t,\ 2,\ 3-4t\rangle$.
- More explicitly, we get $x = 1-2t$, $y = 2$, $z = 3-4t$.

2. The point with $x$-coordinate 6 has $1-2t = 6$ so that $t = -5/2$. Then setting $t = -5/2$ gives the full coordinates as $\langle 6, 2, 13\rangle$.

# Planes in 3-Space, I

Next, we tackle descriptions of planes.

**Proposition (Planes and Normal Vectors)**

*The plane defined by $ax + by + cz = d$ is orthogonal to its normal vector $\mathbf{n} = \langle a, b, c \rangle$. In other words, every line lying in this plane is orthogonal to $\langle a, b, c \rangle$. Conversely, given a nonzero vector $\mathbf{n} = \langle a, b, c \rangle$, there is a unique plane normal to that vector passing through a given point $(x_0, y_0, z_0)$, and its equation is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.*

The key idea here is that we can transfer back and forth between the equation $ax + by + cz = d$ of the plane and the normal vector $\mathbf{n} = \langle a, b, c \rangle$. This transfer is very easy, because we merely have to read off the entries of the normal vector from the coefficients of $x, y, z$, and vice versa.

# Planes in 3-Space, II

Proof:

- Suppose $l$ is a line in the plane. All we need to show is that its direction vector is orthogonal to $\mathbf{n}$.
- So suppose the direction vector is $\mathbf{v} = \mathbf{P}_2 - \mathbf{P}_1$, where both points $\mathbf{P}_2 = \langle x_2, y_2, z_2 \rangle$ and $\mathbf{P}_1 = \langle x_1, y_1, z_1 \rangle$ lie in the plane.
- Then $\mathbf{P}_1 \cdot \mathbf{n} = ax_1 + by_1 + cz_1 = d$ since $\mathbf{P}_1$ lies in the plane, and similarly $\mathbf{P}_2 \cdot \mathbf{n} = d$. But then we have $\mathbf{v} \cdot \mathbf{n} = (\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n} = \mathbf{P}_2 \cdot \mathbf{n} - \mathbf{P}_1 \cdot \mathbf{n} = d - d = 0$, so that $\mathbf{v}$ is orthogonal to $\mathbf{n}$ as claimed.
- For the converse statement, clearly if $\mathbf{n} = \langle a, b, c \rangle$ then the equation of the plane must be $ax + by + cz = \square$ for some value of $\square$, by the previous argument.
- Plugging the given point into the equation shows $\square = ax_0 + by_0 + cz_0$, and finally we can rewrite the equation as $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, as claimed.

# Planes in 3-Space, III

Example: Find an equation for each plane:

1. The plane with normal vector $\langle 2, 1, 4 \rangle$ passing through the point $(2, 2, 0)$.
2. The plane with normal vector $\langle 0, 1, -1 \rangle$ passing through the point $(3, -3, 1)$.

# Planes in 3-Space, III

Example: Find an equation for each plane:

1. The plane with normal vector $\langle 2, 1, 4 \rangle$ passing through the point $(2, 2, 0)$.
2. The plane with normal vector $\langle 0, 1, -1 \rangle$ passing through the point $(3, -3, 1)$.

1. By the proposition, this plane has equation $2(x-2) + 1(y-2) + 4(z-0) = 0$, which we can equivalently write as $2x + y + 4z = 6$.
2. By the proposition, this plane has equation $0(x-3) + 1(y+3) - 1(z-1) = 0$, which we can equivalently write as $y - z = -4$.

# Lines and Planes in 3-Space, I

Using these two results we can solve a wide variety of problems involving lines and planes. The fundamental ideas to remember when working with lines and planes are as follows:

- To specify a line, we need to know its direction vector and a point it passes through.
- To specify a plane, we need to know its normal vector and a point it passes through.

# Lines and Planes in 3-Space, II

Example: Find an equation for the plane containing the vectors $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$ and passing through the point $P = (1, -1, 1)$.

# Lines and Planes in 3-Space, II

Example: Find an equation for the plane containing the vectors $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$ and passing through the point $P = (1, -1, 1)$.

- The normal vector to the plane is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so we can find it by taking their cross product.

# Lines and Planes in 3-Space, II

<u>Example</u>: Find an equation for the plane containing the vectors $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$ and passing through the point $P = (1, -1, 1)$.

- The normal vector to the plane is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so we can find it by taking their cross product.
- We get $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} =$ $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} = \langle 1, -2, 1 \rangle$.
- As a sanity check: $\mathbf{n} \cdot \mathbf{v}_1 = (1)(1) + (-2)(1) + (1)(1) = 0$ and $\mathbf{n} \cdot \mathbf{v}_2 = (1)(1) + (-2)(2) + (1)(3) = 0$.
- Then the plane equation is $1(x-1) - 2(y+1) + 1(z-1) = 0$, or equivalently, $x - 2y + z = 4$.

# Lines and Planes in 3-Space, III

Example: Find an equation for the plane passing through the three points $P_1 = (3, 0, -1)$, $P_2 = (1, 2, 2)$ and $P_3 = (-2, 1, 4)$.

- We need the plane's normal vector $\mathbf{n}$. To find it, we need to find two vectors lying in the plane, and then take their cross product to get $\mathbf{n}$.

# Lines and Planes in 3-Space, III

Example: Find an equation for the plane passing through the three points $P_1 = (3, 0, -1)$, $P_2 = (1, 2, 2)$ and $P_3 = (-2, 1, 4)$.

- We need the plane's normal vector $\mathbf{n}$. To find it, we need to find two vectors lying in the plane, and then take their cross product to get $\mathbf{n}$.
- Two vectors in the plane are $\mathbf{v}_1 = P_2 - P_1$ and $\mathbf{v}_2 = P_3 - P_1$.
- We have $\mathbf{v}_1 = P_2 - P_1 = \langle -2, 2, 3 \rangle$ and $\mathbf{v}_2 = P_3 - P_1 = \langle -5, 1, 5 \rangle$.

# Lines and Planes in 3-Space, III

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- Two vectors in the plane are $\mathbf{v}_1 = P_2 - P_1$ and $\mathbf{v}_2 = P_3 - P_1$.
- We have $\mathbf{v}_1 = P_2 - P_1 = \langle -2, 2, 3 \rangle$ and $\mathbf{v}_2 = P_3 - P_1 = \langle -5, 1, 5 \rangle$.
- Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 2 \\ -5 & 1 \end{vmatrix} \mathbf{k} = \langle 7, -5, 8 \rangle$.
- Sanity check: $\mathbf{n} \cdot \mathbf{v}_1 = (7)(-5) + (-5)(1) + (8)(5) = 0$ and $\mathbf{n} \cdot \mathbf{v}_2 = (7)(-2) + (-5)(2) + (8)(3) = 0$.

# Lines and Planes in 3-Space, IV

Example: Find an equation for the plane passing through the three points $P_1 = (3, 0, -1)$, $P_2 = (1, 2, 2)$ and $P_3 = (-2, 1, 4)$.

- Now we have $\mathbf{n} = \langle 7, -5, 8 \rangle$.
- We can use any of the three points to get the plane's equation. Using $P_1$ yields the equation $7(x - 3) - 5(y - 0) + 8(z + 1) = 0$, or equivalently $7x - 5y + 8z = 13$.
- For an extra error check, we can verify that all three points do lie in this plane: we have $7(3) - 5(0) + 8(-1) = 13$, $7(1) - 5(2) + 8(2) = 13$, and $7(-2) - 5(1) + 8(4) = 13$.
- Thus, this really is the correct equation.

# Lines and Planes in 3-Space, IV

Example: Find an equation for the plane passing through the three points $P_1 = (1, 0, 2)$, $P_2 = (2, 1, 1)$ and $P_3 = (2, 0, 4)$.

# Lines and Planes in 3-Space, IV

Example: Find an equation for the plane passing through the three points $P_1 = (1, 0, 2)$, $P_2 = (2, 1, 1)$ and $P_3 = (2, 0, 4)$.

- Using the method we just described, we compute $\mathbf{v}_1 = P_2 - P_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{v}_2 = P_3 - P_1 = \langle 1, 0, 2 \rangle$.

# Lines and Planes in 3-Space, IV

Example: Find an equation for the plane passing through the three points $P_1 = (1, 0, 2)$, $P_2 = (2, 1, 1)$ and $P_3 = (2, 0, 4)$.

- Using the method we just described, we compute $\mathbf{v}_1 = P_2 - P_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{v}_2 = P_3 - P_1 = \langle 1, 0, 2 \rangle$.
- Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} = \langle 2, -1, -1 \rangle$.
- Then using $P_1$ as our point yields the equation $2(x - 1) - 1(y - 0) - 1(z - 2) = 0$, or equivalently $2x - y - z = 0$.
- It is not hard to see that all three points do satisfy this equation, so this is the correct equation.

## Summary

We discussed cross products and their properties.

We discussed parametrizations of lines and equations of planes in 3-space.

We discussed how to solve various problems involving lines and planes.

Next lecture: More lines and planes in 3-space, vector-valued functions and curves.