Math 2321 (Multivariable Calculus)

Lecture #2 of 38 ∼ January 21, 2021

Vectors and Dot Products

- Vectors and Vector Operations
- Dot Products

This material represents §1.2.1-1.2.2 from the course notes.

Roadmap

Last time, we discussed 3D coordinate system and the associated questions of graphing for functions of 2 or 3 variables.

Today, we will introduce a related topic; namely, vectors and vector operations, which will allow us to develop a slightly different framework in which to view the geometry of curves and surfaces in 3-space.

We will then use the vector framework (next week) to say more things about lines, planes, and curves in 3-dimensional space.

Vectors, I

A vector is a quantity which has both a magnitude and a direction.

- We view a vector in contrast to a scalar, which carries only a magnitude.
- Some examples of vector quantities are force, velocity, acceleration, momentum, torque, and electric and magnetic fields.
- In contrast, quantities like time, distance, mass, and temperature are scalars.

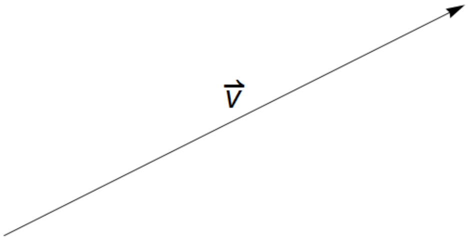
Vectors, II

We denote the n-dimensional vector from the origin to the point $(a_1, a_2, \cdots, a_n)$ as $\mathbf{v} = \langle a_1, a_2, \cdots, a_n \rangle$, where the a_i are scalars.

- We use the angle brackets $\langle \cdot \rangle$ rather than parentheses $(\cdot)$ so as to emphasize the difference between a vector and the coordinates of a point in space. We will, however, view coordinates of vectors and coordinates of points as essentially interchangeable.
- We also write vectors in boldface ($\mathbf{v}$, not v), so that we can tell them apart from scalars. When writing by hand, it is hard to differentiate boldface, so the notation $\vec{v}$ is frequently used.

Vectors, III

We typically think of vectors as directed line segments ("arrows"):



The length of the line segment is the magnitude of the vector, and the direction the segment is pointing is the direction of the vector.

Vectors, IV

As a warning, we remark that vectors are a little bit different from directed line segments, because we don't care where a vector starts: we only care about the difference between the starting and ending positions.

- Thus, the directed segment whose start is $(0, 0)$ and end is $(1, 1)$ and the segment starting at $(1, 1)$ and ending at $(2, 2)$ represent *the same vector* $\langle 1, 1\rangle$.
- This distinction is rarely necessary in most applications, but it is a useful thing to keep in mind when visualizing vectors, since we can view any vector as having any arbitrary starting point we choose.

Vectors, V

We can add vectors (provided they are of the same dimension!) in the obvious way, one component at a time:

Definition

If $\mathbf{v} = \langle a_1, \cdots, a_n \rangle$ *and* $\mathbf{w} = \langle b_1, \cdots, b_n \rangle$ *of the same length, then their* *sum* *is* $\mathbf{v} + \mathbf{w} = \langle a_1 + b_1, \cdots, a_n + b_n \rangle$.

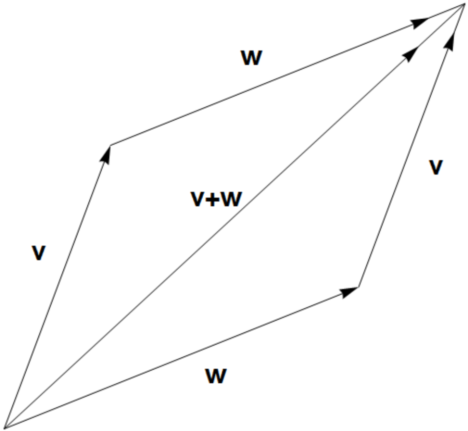
This definition is natural, but we can justify this using our geometric idea of what a vector does:

- Specifically, $\mathbf{v}$ moves us from the origin to $(a_1, \cdots, a_n)$.
- Then $\mathbf{w}$ adds $\langle b_1, \cdots, b_n \rangle$ to the coordinates of our current position, so it moves us from $(a_1, \cdots, a_n)$ to $(a_1 + b_1, \cdots, a_n + b_n)$.
- The net result is that the sum $\mathbf{v} + \mathbf{w}$ moves us from the origin to $(a_1 + b_1, \cdots, a_n + b_n)$, so it is $\langle a_1 + b_1, \cdots, a_n + b_n \rangle$.

Vectors, VI

Geometrically, we can think of vector addition using a parallelogram whose pairs of parallel sides are $\mathbf{v}$ and $\mathbf{w}$ and whose diagonal is $\mathbf{v} + \mathbf{w}$, which also explains that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$:

Parallelogram for Vector Addition



Vectors, VII

We can also 'scale' a vector by a scalar, one component at a time:

Definition

If r is a scalar and $\mathbf{v} = \langle a_1, \dots, a_n \rangle$ is a vector, then the scalar multiple $r\mathbf{v}$ is defined as $r\,\mathbf{v} = \langle ra_1, \cdots, ra_n \rangle$.

Again, we can justify this by our geometric idea of what a vector does:

- If $\mathbf{v}$ moves us some amount in a direction, then $\dfrac{1}{2}\mathbf{v}$ should move us half as far in that direction.
- Analogously, $2\mathbf{v}$ should move us twice as far in that direction, while $-\mathbf{v}$ should move us exactly as far, but in the opposite direction.

Vectors, VIII

Example: If $\mathbf{v} = \langle -1, 2, 2 \rangle$ and $\mathbf{w} = \langle 3, 0, -4 \rangle$, find the following:

1. $2\mathbf{w}$.
2. $\mathbf{v} + \mathbf{w}$.
3. $\mathbf{v} - 2\mathbf{w}$.

Vectors, VIII

Example: If $\mathbf{v} = \langle -1, 2, 2 \rangle$ and $\mathbf{w} = \langle 3, 0, -4 \rangle$, find the following:

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3. $\mathbf{v} - 2\mathbf{w}$.

1. We have $2\mathbf{w} = \langle 2 \cdot 3, 2 \cdot 0, 2 \cdot (-4) \rangle = \langle 6, 0, -8 \rangle$.
2. Also, $\mathbf{v} + \mathbf{w} = \langle -1 + 3, 2 + 0, 2 + (-4) \rangle = \langle 2, 2, -2 \rangle$.
3. Finally, $\mathbf{v} - 2\mathbf{w} = \langle -1, 2, 2 \rangle - \langle 6, 0, -8 \rangle = \langle -7, 2, 10 \rangle$.

Vectors, IX

Arithmetic of vectors satisfies several algebraic properties that follow from the definition:

1. Addition of vectors is commutative: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
2. Addition of vectors is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
3. There is a zero vector $\mathbf{0}$ (namely, the vector with all entries zero) such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for every $\mathbf{v}$.
4. Every vector $\mathbf{v}$ has an additive inverse $-\mathbf{v}$ (namely, the vector with all entries scaled by -1) such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. Scalar multiplication distributes over addition of vectors ($r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$) and scalars ($(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$).
6. Scaling is consistent ($(ab)\mathbf{v} = a(b\mathbf{v})$) and 1 is an identity ($1\mathbf{v} = \mathbf{v}$).

In linear algebra (if you have not already taken it) you will learn about general vector spaces, which abstract the concept of a vector from the properties listed here.

Vectors, X

At this point, we will restrict ourselves to talking just about 2-dimensional space and 3-dimensional space.

- Our primary reason for this is that most of the immediate applications of vectors (e.g., to physics) happen in 3-dimensional space.
- If you want to learn about higher-dimensional things, take more advanced math courses such as linear algebra and real analysis.

Vectors, XI

It will be useful to have a way to denote the "unit coordinate" vectors of 3-dimensional space.

Definition

The unit coordinate vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *are* $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, *and* $\mathbf{k} = \langle 0, 0, 1 \rangle$.

- We can rewrite any vector in 2-space or 3-space as a linear combination of the unit coordinate vectors.
- Example: The vector $\langle 3, 2, -5 \rangle$ is equal to $3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.
- Note: Some authors primarily use **ijk**-notation when working with vectors. We will generally use the angle bracket notation, except in a few cases when there are useful mnemonics that are easier to remember using **ijk**-notation. But you will see such notation in the WeBWorK problems (this is to make sure you're comfortable with what you might see elsewhere!)

Lengths, I

We now analyze lengths and angles between vectors.

Definition

We define the *norm* *(also called the* *length* *or* *magnitude*) *of the vector* $\mathbf{v} = \langle a_1, \ldots, a_n \rangle$ *as* $||\mathbf{v}|| = \sqrt{(a_1)^2 + \cdots + (a_n)^2}$.

- This is just an application of the distance formula: the norm of the vector $\langle a_1, \ldots, a_n \rangle$ is just the length of the line segment joining the origin $(0, \ldots, 0)$ to the point $(a_1, \ldots, a_n)$.

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- Example: For $\mathbf{v} = \langle -1, 2, 2 \rangle$ and $\mathbf{w} = \langle 3, -4 \rangle$, we have $||\mathbf{v}|| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$ and $||\mathbf{w}|| = \sqrt{3^2 + (-4)^2} = 5$.
- If r is a scalar, we can see immediately from the definition that $||r\,\mathbf{v}|| = |r|\ ||\mathbf{v}||$, since we can just factor out a $\sqrt{r^2} = |r|$ from each term under the square root.

Lengths, II

Starting with any nonzero vector, we can construct a unit vector (that is, a vector of length 1) in the same direction of $\mathbf{v}$ just by scaling $\mathbf{v}$ by 1 over its length.

Definition

If $\mathbf{v}$ is a nonzero vector, the vector $\mathbf{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||}$, called the <u>normalization</u> of $\mathbf{v}$, is a unit vector in the same direction as $\mathbf{v}$.

<u>Example</u>: Find a unit vector in the same direction of $\mathbf{v} = \langle -1, 2, 2 \rangle$ and of $\mathbf{w} = \langle 3, -4 \rangle$.

Lengths, II

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<u>Example</u>: Find a unit vector in the same direction of $\mathbf{v} = \langle -1, 2, 2 \rangle$ and of $\mathbf{w} = \langle 3, -4 \rangle$.

- We compute $||\mathbf{v}|| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$, so a unit vector in the same direction as $\mathbf{v}$ is $\mathbf{u}_1 = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \left\langle -\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3} \right\rangle$.
- Likewise, we have $||\mathbf{w}|| = \sqrt{3^2 + (-4)^2} = 5$, so $\mathbf{u}_2 = \left\langle \dfrac{3}{5}, -\dfrac{4}{5} \right\rangle$ is a unit vector in the direction of $\mathbf{w}$.

The Dot Product, I

Another thing we might want to know about two vectors is the angle θ between them. This (admittedly anemic) question is our motivation for defining the dot product of two vectors:

Definition

The dot product of two vectors $\mathbf{v}_1 = \langle a_1, \ldots, a_n \rangle$ *and* $\mathbf{v}_2 = \langle b_1, \ldots, b_n \rangle$ *is defined to be the scalar* $\mathbf{v}_1 \cdot \mathbf{v}_2 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

Note that the dot product of two vectors is a scalar, NOT a vector! For this reason, the dot product is sometimes called the scalar product.

The Dot Product, II

Examples: Find each of the following dot products:

1. $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$.
2. $\langle -1, 2, 2 \rangle \cdot \langle 3, 0, -4 \rangle$.
3. $\langle 5, 6 \rangle \cdot \langle -6, 5 \rangle$.
4. $\langle -1, 2, 2 \rangle \cdot \langle 3, 4 \rangle$.

The Dot Product, II

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2. $\langle -1, 2, 2 \rangle \cdot \langle 3, 0, -4 \rangle$.
3. $\langle 5, 6 \rangle \cdot \langle -6, 5 \rangle$.
4. $\langle -1, 2, 2 \rangle \cdot \langle 3, 4 \rangle$.

- $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$ is $(1)(3) + (2)(4) = 11$.
- $\langle -1, 2, 2 \rangle \cdot \langle 3, 0, -4 \rangle$ is $(-1)(3) + (2)(0) + (2)(-4) = -11$.
- $\langle 5, 6 \rangle \cdot \langle -6, 5 \rangle$ is $(5)(-6) + (6)(5) = 0$.
- The dot product $\langle -1, 2, 2 \rangle \cdot \langle 3, 4 \rangle$ does not make sense: the vectors are not the same length.

The Dot Product, III

The dot product possesses several numerous properties reminiscent of standard multiplication.

Proposition (Properties of Dot Products)

For any vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$*, and any scalar* r*, the following properties hold:*

1. *The dot product distributes over addition:* $(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = (\mathbf{v}_1 \cdot \mathbf{w}) + (\mathbf{v}_2 \cdot \mathbf{w})$.
2. *The dot product distributes through scaling:* $(r\mathbf{v}) \cdot \mathbf{w} = r(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (r\mathbf{w})$.
3. *The dot product is commutative:* $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
4. *The dot product of a vector with itself is the square of the norm:* $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$. *(In particular,* $\mathbf{v} \cdot \mathbf{v} \geq 0$ *for all vectors* $\mathbf{v}$*.)*

<u>Proofs</u>: Each of these properties is a simple algebraic calculation from the definition of the dot product.

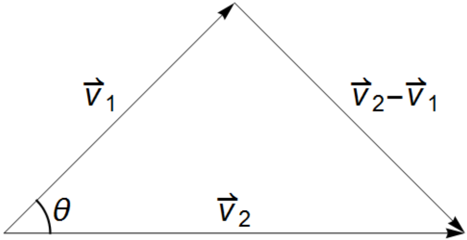
The Dot Product, IV

There is a nice relation between the dot product and the angle between two vectors:

Theorem (Dot Product Theorem)

For vectors $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *forming an angle* θ *between them, we have* $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \; ||\mathbf{v}_2|| \cos(\theta)$.

Proof: Apply the Law of Cosines in the triangle formed by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2 - \mathbf{v}_1$:



The Dot Product, V

<u>Proof</u> (continued):

- We get $||\mathbf{v}_2 - \mathbf{v}_1||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2\,||\mathbf{v}_1||\,||\mathbf{v}_2||\cos(\theta)$.
- Since $||\mathbf{w}||^2 = \mathbf{w} \cdot \mathbf{w}$ for any vector $\mathbf{v}$, we can then convert the statement above to one involving dot products, and then apply the various properties of dot products:

$$\begin{aligned} ||\mathbf{v}_2 - \mathbf{v}_1||^2 &= (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) \\ &= (\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2) - (\mathbf{v}_2 \cdot \mathbf{v}_1) + (\mathbf{v}_1 \cdot \mathbf{v}_1) \\ &= ||\mathbf{v}_2||^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) + ||\mathbf{v}_1||^2 . \end{aligned}$$

- Then, by comparing the expression above to the Law of Cosines expression and cancelling terms, we are left with $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1||\ ||\mathbf{v}_2||\ \cos(\theta)$, as desired.

The Dot Product, VI

Example: Compute the angle between the vectors $\mathbf{v} = \langle 2, 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 0, \sqrt{3}, 1 \rangle$.

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- We compute $\mathbf{v} \cdot \mathbf{w} = (2)(0) + (1)(\sqrt{3}) + (\sqrt{3})(1) = 2\sqrt{3}$, and $||\mathbf{v}|| = \sqrt{2^2 + 1^2 + (\sqrt{3})^2} = \sqrt{8} = 2\sqrt{2}$ and $||\mathbf{w}|| = \sqrt{(\sqrt{3})^2 + 0^2 + 1^2} = 2$.

The Dot Product, VI

Example: Compute the angle between the vectors $\mathbf{v} = \langle 2, 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 0, \sqrt{3}, 1 \rangle$.

- We compute $\mathbf{v} \cdot \mathbf{w} = (2)(0) + (1)(\sqrt{3}) + (\sqrt{3})(1) = 2\sqrt{3}$, and $||\mathbf{v}|| = \sqrt{2^2 + 1^2 + (\sqrt{3})^2} = \sqrt{8} = 2\sqrt{2}$ and $||\mathbf{w}|| = \sqrt{(\sqrt{3})^2 + 0^2 + 1^2} = 2$.
- Then by the Dot Product Theorem, the angle θ between the vectors satisfies $2\sqrt{3} = 2 \cdot 2\sqrt{2} \cdot \cos(\theta)$, meaning that $\theta = \cos^{-1}\left(\sqrt{3/8}\right) \approx 0.91$ radians.

The Dot Product, VII

Example: Compute the angle between the vectors $\mathbf{v} = \langle 2, 2, -1 \rangle$ and $\mathbf{w} = \langle 3, 4, 0 \rangle$.

The Dot Product, VII

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- We compute $\mathbf{v} \cdot \mathbf{w} = (2)(3) + (2)(4) + (-1)(0) = 14$, and $||\mathbf{v}|| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$ and $||\mathbf{w}|| = \sqrt{3^2 + (4)^2 + 0^2} = 5$.
- Then by the Dot Product Theorem, the angle θ between the vectors satisfies $14 = 3 \cdot 5 \cdot \cos(\theta)$, so $\theta = \cos^{-1}\left(\dfrac{14}{15}\right) \approx 0.3672 \approx 21.04^\circ$.

The Dot Product, VIII

Using the Dot Product Theorem, we can see that the sign and magnitude of the dot product is (roughly) measuring whether the vectors are pointing in the same direction:

- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero, both $||\mathbf{v}_1||$ and $||\mathbf{v}_2||$ are positive.
- Thus, if we examine the Dot Product Theorem $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \; ||\mathbf{v}_2|| \cos(\theta)$, the value $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \; ||\mathbf{v}_2|| \cos(\theta)$ will have the same sign as $\cos(\theta)$.

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Using the Dot Product Theorem, we can see that the sign and magnitude of the dot product is (roughly) measuring whether the vectors are pointing in the same direction:

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- Thus, if we examine the Dot Product Theorem $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \; ||\mathbf{v}_2|| \cos(\theta)$, the value $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \; ||\mathbf{v}_2|| \cos(\theta)$ will have the same sign as $\cos(\theta)$.
- If $0 \leq \theta < \pi/2$, the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be positive. Furthermore, the smaller θ is, the larger the value of $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be. Thus, a large positive value for the dot product indicates that the vectors are pointing in roughly the same direction.
- Inversely, if $\pi/2 < \theta \leq \pi$, the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be negative, and the larger θ is, the larger negative $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be. Thus, a large negative value for the dot product indicates that the vectors are pointing in roughly opposite directions.

The Dot Product, IX

We have a special name for the case where the angle between two vectors is $\pi/2$:

Definition

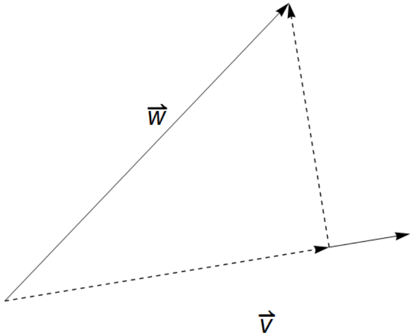
We say two vectors are <u>orthogonal</u> if their dot product is zero.

- From the Dot Product Theorem, since $\cos(\pi/2) = 0$, we see that two nonzero vectors are orthogonal if the angle between them is $\pi/2$, which is to say, if they are perpendicular.
- <u>Example</u>: The vectors $\langle 2, -1, 4 \rangle$ and $\langle 3, 2, -1 \rangle$ are orthogonal, since their dot product is $(2)(3) + (-1)(2) + (4)(-1) = 0$.
- <u>Remark</u>: Since the dot product of the zero vector with any vector is zero, by our definition above, the zero vector is orthogonal to every vector.

Vector Projection, I

Another basic question about vectors is: given a vector **v** and another vector **w**, how much of **w** is in the direction of **v**, and how much of **w** is orthogonal to **v**?

- This problem often arises in Newtonian physics: one often needs to separate the vector representing a force (or velocity, or acceleration, ...) into orthogonal components.



Vector Projection, II

We want to write $\mathbf{w} = a\mathbf{v} + \mathbf{y}$, where $\mathbf{y}$ is orthogonal to $\mathbf{v}$. The goal is to determine the value of the scalar a.

- We can rewrite the expression as $\mathbf{y} = \mathbf{w} - a\mathbf{v}$, where by assumption $\mathbf{y}$ is orthogonal to $\mathbf{v}$.
- But now taking the dot product of both sides with $\mathbf{v}$ and applying the algebraic properties of dot products yields $0 = \mathbf{y} \cdot \mathbf{v} = (\mathbf{w} - a\mathbf{v}) \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v} - a(\mathbf{v} \cdot \mathbf{v})$.
- Now it is easy to solve for a: doing so yields $a = \dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$.

Vector Projection, III

We summarize the previous calculations as follows:

Definition

If $\mathbf{v}$ is a nonzero vector and $\mathbf{w}$ is any vector, then the *<u>vector projection of $\mathbf{w}$ onto $\mathbf{v}$</u> is the vector* $Proj_{\mathbf{v}}(\mathbf{w}) = \left(\dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$.

- The vector projection of $\mathbf{w}$ onto $\mathbf{v}$ gives the "piece" of $\mathbf{w}$ in the direction of $\mathbf{v}$, while the remaining piece $\mathbf{w} - \text{Proj}_{\mathbf{v}}(\mathbf{w})$ is orthogonal to $\mathbf{v}$.

Vector Projection, IV

Example: Find the vector projection of $\mathbf{w} = \mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ onto the coordinate vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Vector Projection, IV

<u>Example</u>: Find the vector projection of $\mathbf{w} = \mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ onto the coordinate vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

- We have $\text{Proj}_{\mathbf{i}}(\mathbf{w}) = \dfrac{\mathbf{w} \cdot \mathbf{i}}{\mathbf{i} \cdot \mathbf{i}}\mathbf{i} = \dfrac{1}{1}\mathbf{i} = 1\mathbf{i}$.
- Next, $\text{Proj}_{\mathbf{j}}(\mathbf{w}) = \dfrac{\mathbf{w} \cdot \mathbf{j}}{\mathbf{j} \cdot \mathbf{j}}\mathbf{j} = \dfrac{6}{1}\mathbf{j} = 6\mathbf{j}$.
- Finally, $\text{Proj}_{\mathbf{k}}(\mathbf{w}) = \dfrac{\mathbf{w} \cdot \mathbf{k}}{\mathbf{k} \cdot \mathbf{k}}\mathbf{k} = \dfrac{5}{1}\mathbf{k} = 5\mathbf{k}$.

You should find these results very consistent with what the vector projection actually represents.

- For example, the component of $\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$ in the direction of $\mathbf{j}$ should logically be 6, and this is exactly what we found.

Vector Projection, V

Example: Find the vector projection of $\mathbf{w} = \langle 6, -3, 0 \rangle$ onto $\mathbf{v} = \langle 2, -2, -1 \rangle$, and verify that $\mathbf{w} - \mathrm{Proj}_{\mathbf{v}}(\mathbf{w})$ is orthogonal to $\mathbf{v}$.

Vector Projection, V

Example: Find the vector projection of $\mathbf{w} = \langle 6, -3, 0 \rangle$ onto $\mathbf{v} = \langle 2, -2, -1 \rangle$, and verify that $\mathbf{w} - \text{Proj}_{\mathbf{v}}(\mathbf{w})$ is orthogonal to $\mathbf{v}$.

- We have $\text{Proj}_{\mathbf{v}}(\mathbf{w}) = \left(\dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \dfrac{(6)(2) + (-3)(-2) + (0)(-1)}{2^2 + (-2)^2 + (-1)^2} \mathbf{v} = \dfrac{18}{9} \mathbf{v} = 2\mathbf{v} = \langle 4, -4, -2 \rangle$.
- To check the orthogonality, note that $\mathbf{w} - \text{Proj}_{\mathbf{v}}(\mathbf{w}) = \langle 2, 1, 2 \rangle$.
- Indeed, $\langle 2, 1, 2 \rangle$ is orthogonal to $\mathbf{v}$, because $\langle 2, 1, 2 \rangle \cdot \langle 2, -2, -1 \rangle = (2)(2) + (1)(-2) + (2)(-1) = 0$.

Summary

We discussed vectors and vector operations.

We discussed lengths of vectors, dot products, angles between vectors, and vector projection.

Next lecture: Cross products, lines and planes in 3-space.