Math 2321 (Multivariable Calculus)

Lecture #1 of 38 ~ January 20, 2021

Welcome + 3D Coordinates

- Welcome + Logistics
- 3D Coordinates

This material represents §1.1 from the course notes.

Welcome!

Welcome to Math 2321 (Multivariable Calculus)! Here are some course-related locations to bookmark:

- The course webpage is here: `https://web.northeastern.edu/dummit/teaching_sp21_2321.html` . Most course-related information is posted there.
- Homework assignments are in WeBWorK: `https://courses1.webwork.maa.org/webwork2/northeastern-math2321/` .
- Course-related discussion will be done via Piazza: `https://piazza.com/class/kj7dka0nnpk24m` .
- Exams will be distributed and collected via Canvas: `https://canvas.northeastern.edu/` .

Course Topics: Multivariable Calculus

As you might expect based on the title, this course is about multivariable calculus. Roughly speaking, we redo the material from Calculus 1 + 2 in a multidimensional setting, in 4 segments:

1. Vectors and 3D Geometry: 3D coordinates and graphing, vectors, dot and cross products, lines and planes, vector-valued functions, curves and motion in 3-space.
2. Partial Derivatives: Limits, partial derivatives, directional derivatives, gradients, tangent planes, the chain rule, linearization, critical points (minima and maxima), Lagrange multipliers.
3. Multiple Integration: Double integrals in rectangular and polar coordinates, coordinate changes, triple integrals in rectangular, cylindrical, and spherical coordinates, applications of multiple integration.
4. Vector Calculus: Line integrals, surfaces and surface integrals, vector fields, work, circulation, flux, conservative vector fields and potential functions, Green's theorem, divergence and curl, Gauss's divergence theorem, Stokes's theorem, applications of vector calculus.

Lectures

The course lectures will be conducted via Zoom. All lectures are recorded for later viewing. For security reasons (since these lecture slides are posted publicly) the links to upcoming and past lectures are only available via the Canvas page or via the Piazza page.

- Section 4 meets MWR from 1:35pm-2:40pm Eastern time.
- Section 12 meets MWR from 4:35pm-5:40pm Eastern time.

Lecture attendance is not required. For your convenience, you may attend either lecture in case one works better for your schedule than the other (they will cover the same material at the same pace).

Grades, I

Your course grade consists of 16% WeBWorK (homework) and 84% exams. More specifically:

- There will be 12 WeBWorK assignments, one each week. Every problem on every assignment will be counted, and your score is [total points scored]/[total problems assigned].
- There will be 4 exams: three one-hour midterms and a cumulative final. If you miss an exam for any reason, you will receive a zero. Makeup exams will not be given.
- Each midterm contributes 18% of your course grade and the common final contributes 30%. The lowest midterm grade can be half-dropped (thus contributing 9%) and replaced with the score on the final exam, if this would increase your grade. This is an automatic calculation at the end of the semester.

Grades, II

The letter grades in the course are not assigned via a fixed scale (it is determined by the distribution at the end of the term). However, there are lower bounds:

- A raw average of 92% will be at least an A.
- A raw average of 90% will be at least an A-.
- A raw average of 88% will be at least a B+.
- A raw average of 82% will be at least a B.
- A raw average of 80% will be at least a B-.
- A raw average of 78% will be at least a C+.
- A raw average of 70% will be at least a C.
- A raw average of 68% will be at least a C-.
- A raw average of 60% will guarantee a passing grade (i.e., D).

Approximate grade correspondences will be provided with the midterm exam scores so that you can track your progress.

Exams

There are four exams in this course: three midterm exams and a cumulative final exam.

- The three midterms will be held on Fri Feb 19th, Fri Mar 19th, and Tue Apr 13th via Canvas. You will have an option of at least four possible time windows, spread throughout the day (also including the next day), for maximum flexibility.
- The final exam will be held during finals week (date TBA).
- The exams will have the same format and length as if they were being given in class. You will sign up for a fixed time window in which to take your exam and submit scans/photos of your responses.

I will post ample review material ahead of the exams, and I plan to hold at least one (typically two) in-class review sessions ahead of each exam.

WeBWorK, I

There will be 12 WeBWorK assignments, one each week, typically due at 5am Eastern on Thursdays.

- WeBWorK is an open-source electronic homework system designed specifically for mathematics and statistics courses. The main advantage (for you) is that it is free for students.
- Problems are randomized, so that each of you will have a different version of the problem. This allows you to work together without having the learning spoiled.
- Each assignment will have approximately 20 problems. All problems are worth 1 point; many problems have multiple parts, each of which is worth some fraction of 1 point.
- You are encouraged to consider the homeworks as being due "Wednesday evening".

WeBWorK, II

To access WeBWorK, go to the WeBWorK URL and enter your username and password.

- Your WeBWorK username is your NU login name (typically formed from a combination of your first and last names, with no spaces or hyphens). This is the same as the portion of your NEU Husky email address before the "@" symbol.
- Your WeBWorK password is your NU ID number without the two leading zeroes. It will be 7 digits in length.
- Example: Someone named Cool Person with ID number 009876543 and email address person.c@husky.neu.edu would have WeBWorK username person.c and password 9876543 .

Try logging into WeBWorK now to check that things work properly, if you haven't already done so.

WeBWorK, III

How to use WeBWorK as wisely as possible:

- Consider the homeworks as being due "Wednesday evening".
- Look over the problems as soon as the set is open so you know what material is covered, and then spread out your work on the problems over several days. Do not fall into the trap of only starting the assignment the evening before it is due!
- If you are stuck on a problem, move to another one and then come back. If you're still stuck, ask for help on Piazza, during office hours, or using WeBWorK's "Email Instructor" button.
- Make sure to do as much of every assignment as possible, even if you cannot completely finish it.

Remark: I have taught multivariable calculus five times. In every one of those courses, the only students who *ever* did poorly were those who didn't do the WeBWorK problems every week!

WeBWorK, IV

You are allowed THREE 24-hour WeBWorK extensions during the semester. Additional extensions will not be granted under any circumstances.

- To claim an extension on a set, email the instructor within 24 hours of the due date requesting to use your extension.
- The 24-hour extension applies to the original due date, not to the time you request the extension.
- Please make requests as follows: "I would like to use a 24-hour extension on Set #" (please include the set number!).
- You need not give any reason for requesting an extension.
- You may apply more than one extension to the same set (using 2 gives 48 hours, using all 3 gives 72 hours), but you only get 3 total. Unused extensions will be automatically applied to the last set of the semester.

Piazza

We will use Piazza as a course discussion forum: `https://piazza.com/class/kj7dka0nnpk24m`. The Piazza page is open only by invitation to 2321 students. Check your email for the invitation.

- Links to the recordings of the past lectures will be hosted on the Piazza page, as well as the link to upcoming lectures and problem sessions.
- You are encouraged to conduct discussions about the course and associated topics on Piazza. In particular, you are welcome to ask questions about the WeBWorK problems. If you do, it is helpful to others if you include a snapshot of your version of the problem.

Try visiting the Piazza page now, if you haven't already done so.

Office Hours

I will hold weekly office hours, intended as a time for you to get one-on-one help with WeBWorK or ask other course-related questions. Office hours are at the following times:

- Wednesday 3:00pm-4:15pm + Thursday 12:15pm-1:15pm

Office hours are held via Zoom and are not recorded.

- By default, office hours are set up as a waiting room to allow one-on-one discussion.
- However, if the waiting list becomes long, I may switch format to an open group discussion.
- If you wish to discuss matters privately during an open discussion, please send me a private chat message and I will reserve time to talk to you.

Miscellaneous Info

Here is some other miscellaneous information:

- The instructor will write lecture notes for the course (in lieu of an official textbook) as the semester progresses. The course will roughly follow the presentation in Massey's "Worldwide Multivariable Calculus", but it is not necessary to purchase the textbook. (Essentially all calculus books are equivalent and cover the same material in the same order.)
- Course prerequisites: Math 1242 or 1252 or 1342 (Calculus 2).
- Collaboration and technology: You are free to use calculators and computer technology for homework problems, and calculators are allowed on exams. You are allowed to work on, and discuss, homework assignments together, as long as the actual submissions are your own work. Collaboration is, of course, not allowed on exams.

Other Boilerplate, I

- Statement on Academic Integrity: A commitment to the principles of academic integrity is essential to the mission of Northeastern University. Academic dishonesty violates the most fundamental values of an intellectual community and undermines the achievements of the entire University. Violations of academic integrity include (but are not limited to) cheating on assignments or exams, fabrication or misrepresentation of data or other work, plagiarism, unauthorized collaboration, and facilitation of others' dishonesty. Possible sanctions include (but are not limited to) warnings, grade penalties, course failure, suspension, and expulsion.

Other Boilerplate, II

- Statement on Accommodations: Any student with a disability is encouraged to meet with or otherwise contact the instructor during the first week of classes to discuss accommodations. The student must bring a current Memorandum of Accommodations from the Office of Student Disability Services.
- Statement on Classroom Behavior: Disruptive classroom behavior will not be tolerated. In general, any behavior that impedes the ability of your fellow students to learn will be viewed as disruptive.
- Statement on Inclusivity: Faculty are encouraged to address students by their preferred name and gender pronoun. If you would like to be addressed using a specific name or pronoun, please let your instructor know.

Other Boilerplate, III

- Statement on Evaluations: Students are requested to complete the TRACE evaluations at the end of the course.
- Miscellaneous Disclaimer: The instructor reserves the right to change course policies, including the evaluation scheme of the course (e.g., in the event of natural disaster or global pandemic). Notice will be given in the event of any substantial changes.

Awkward Transition To Doing Actual Math Now

Pause here for questions about course logistics.

Note to self: don't read this slide out loud.

Overview of §1: Vectors and 3D Geometry

Our first topic is to discuss functions of several variables, vectors, and 3D geometry.

- We begin with a brief discussion of 3D coordinates and graphing. (Pictures!)
- Then we discuss vectors and vector operations, and finish with some of their applications to 3D graphing and geometry.

Functions of Several Variables, I

In one-variable calculus (i.e., calculus 1 and 2), we spend most of our time analyzing 2-dimensional geometry. Some examples of such things:

- Graphing a curve of the form $y = f(x)$ and identifying interesting features (e.g., the slope of the tangent line, minimum and maximum points, calculating the area under the curve, etc.).
- Graphing a parametric curve, such as $x = t + \sin(t)$, $y = 1 + \cos(t)$ and identifying interesting features (e.g., slope of the tangent line, area inside the curve).
- Graphing an implicitly-defined function, such as $x^2 + y^2 = 1$ and identifying some features (e.g., slope of the tangent line).

The primary theme of the course is to generalize these things to higher dimensions.

Functions of Several Variables, II

Our first step is to discuss functions of several variables, which are functions that take in several variables and output a unique, well-defined value associated to the inputs.

Examples:

- The function $f(x, y) = x + y$ takes in two values x and y and outputs their sum $x + y$.
- The function $g(x, y, z) = xyz^2$ takes in three values x, y, and z, and outputs the product xyz^2.
- The function $d(l, w) = \sqrt{l^2 + w^2}$ gives the length of a diagonal of a rectangle as a function of the rectangle's length l and its width w.
- The function $V(r, h) = \frac{1}{3}\pi r^2 h$ gives the volume of a (right circular) cone in terms of its base radius r and its height h.

Functions of Several Variables, III

As with functions of one variable, a function of several variables has a domain and a range: the domain is the set of input values and the range is the set of output values.

- For a function of two variables $f(x, y)$, the domain is now a subset of the 2-dimensional plane rather than a subset of the real line. Because of this, domains of functions of more than one variable can be rather complicated.
- In general, unless specified, the domain of a function is the largest possible set of inputs for which the definition of the function makes sense.
- We adopt the convention that our functions are assumed to be real-valued unless otherwise specified, so that square roots of negative real numbers are not allowed, nor is division by zero.
- The range of a function of several variables will still be a subset of the real line.

Functions of Several Variables, IV

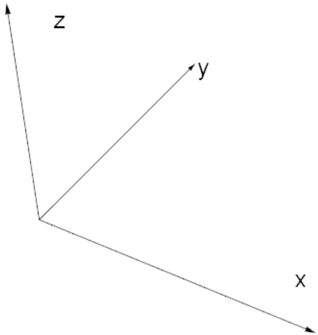
<u>Examples</u>:

- For $f(x, y) = \sqrt{x} + \sqrt{y}$, the domain is the first quadrant of the xy-plane, defined by the inequalities $x \geq 0$ and $y \geq 0$. The range is the set $[0, \infty)$.
- For $g(x, y) = \sqrt{x^2 + y^2 - 1}$, the domain is the set of points in the xy-plane which satisfy $x^2 + y^2 \geq 1$: this describes all the points of the plane except for those lying strictly inside the unit circle. The range is $[0, \infty)$.
- For $h(x, y) = \dfrac{1}{x - y}$, the domain is the set of points in the xy-plane which satisfy $x - y \neq 0$: this describes all points in the plane except those on the line $y = x$. The range is all nonzero real numbers.

3-Space, I

In order to graph functions of several variables, we need to do things in 3-dimensional space, which we call 3-space for short.

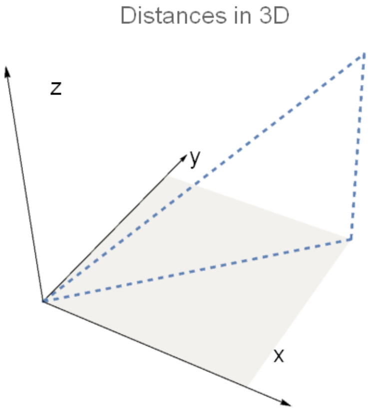
- Points in 3-space are represented by a triplet of numbers (x, y, z). The new coordinate z represents height above the xy-plane.



3-Space, II

Algebraically, 3-space behaves quite similarly to 2-space.

- For example, by two applications of the Pythagorean Theorem, the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.



Graphing Functions of 2 Variables

There are two primary ways to visualize a function $f(x, y)$ of two variables.

- The first way is to plot the points (x, y, z) in 3-space satisfying $z = f(x, y)$.
- At the point (x, y) in the plane, the graph has the height $z = f(x, y)$; so we see that as (x, y) varies through the plane, the function $z = f(x, y)$ will trace out a surface, called the graph of $f(x, y)$.
- The second way is to plot level sets, which we discuss more in a moment.

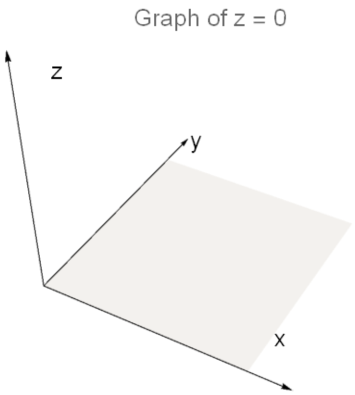
Technology and Graphing

In the next slides, I will include still plots of 3D graphs of these various surfaces.

- However, it is much easier to get a sense of what these 3D graphs look like if you use software that allows you to manipulate the plots and change the viewing angle.

Examples of Graphs, I

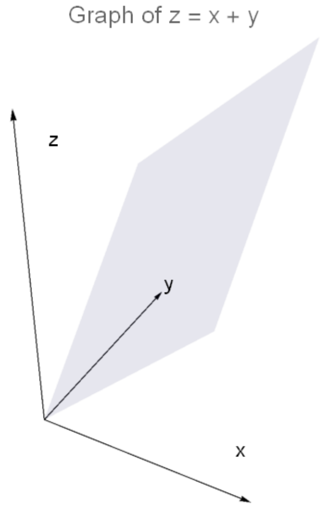
Example: The graph $z = 0$ is the xy-plane.



More generally, any equation of the form $z = ax + by + d$ will give a plane. (We will discuss equations of planes next week.)

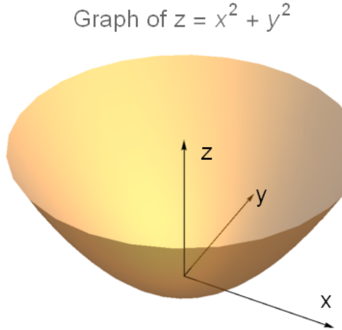
Examples of Graphs, II

Example: The graph $z = x + y$ is a tilted plane.



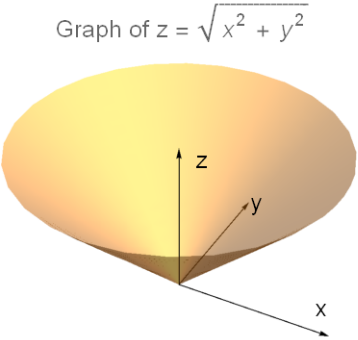
Examples of Graphs, III

Example: The graph $z = x^2 + y^2$ is a parabolic dish. The more formal name is a paraboloid.



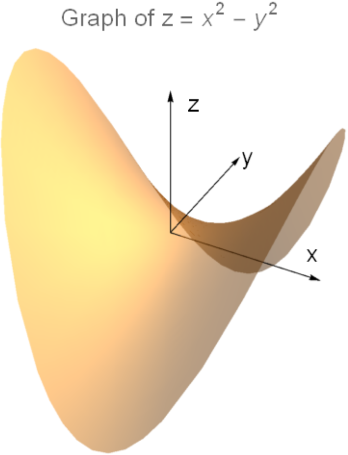
Examples of Graphs, IV

Example: The graph $z = \sqrt{x^2 + y^2}$ is a right circular cone opening upward, with vertex at the origin.



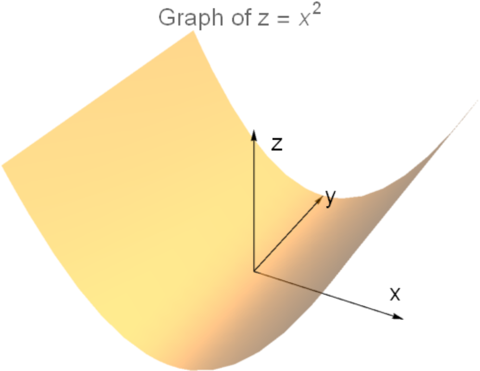
Examples of Graphs, V

<u>Example</u>: The graph $z = x^2 - y^2$ is a <u>saddle surface</u>, also called (more formally) a <u>hyperbolic paraboloid</u>.



Examples of Graphs, VI

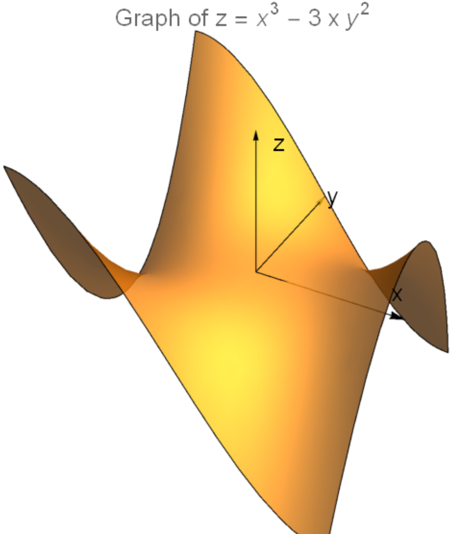
Example: The graph $z = x^2$ is a parabolic cylinder.



In general, any surface obtained by translating a plane curve along an axis is called a cylinder. This cylinder is obtained by translating the plane curve $z = x^2$ along the y-axis.

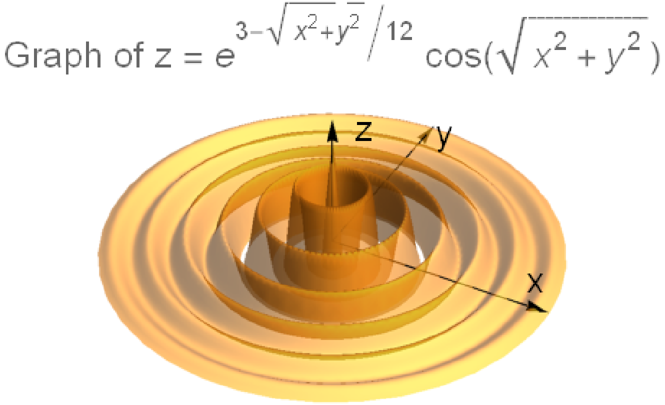
Examples of Graphs, VII

Example: Here's the graph of $z = x^3 - 3xy^2$.



Examples of Graphs, VIII

Example: Here's the graph of $z = e^{3-\sqrt{x^2+y^2}/12} \cdot \cos\left(\sqrt{x^2+y^2}\right)$.



Examples of Graphs, IX

Example: [Pick some other function of your own choosing, and have a computer plot the graph.]

- I use Mathematica to produce most of the graphics in this course. Mathematica is available free for all Northeastern students. One nice feature about Mathematica is that it allows you to rotate and manipulate 3D graphics by default, so you can move the pictures around.
- Other popular choices are Desmos, GeoGebra, or typing “3d grapher” into Google.
- I highly encourage you to find a grapher you like and to get comfortable using it.

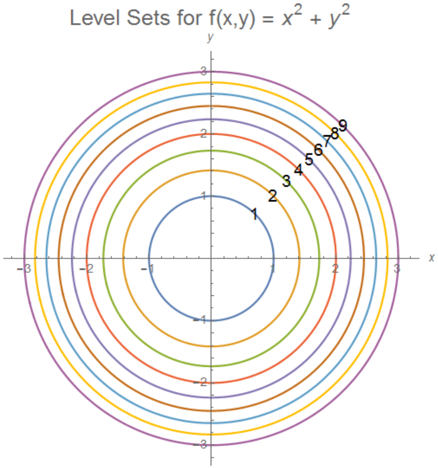
Level Sets, I

The second way to visualize a function $f(x, y)$ is is to plot the points (x, y) in the plane on the level sets $f(x, y) = c$ for particular values of c, as implicit curves.

- For a given function $f(x, y)$ and a particular value of c, the points (x, y) satisfying $f(x, y) = c$ are called a level set of f.
- For a function of two variables these sets will generally be curves, so they are also sometimes called level curves.
- Level sets are obtained by intersecting the horizontal plane $z = c$ with the graph of the surface $z = f(x, y)$.
- If we graph many of these level curves together on the same axes, we will obtain a "topographical map" of the function $f(x, y)$.

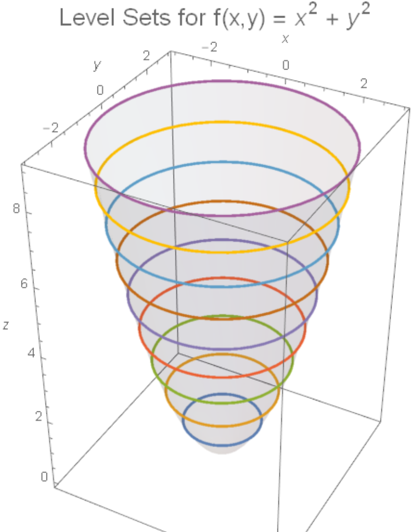
Level Sets, II

Example: The level sets for the function $f(x, y) = x^2 + y^2$ are circles in the plane.



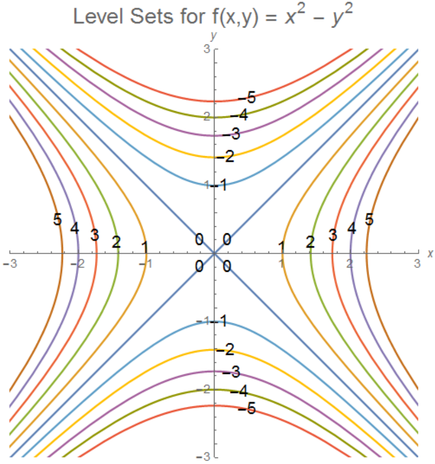
Level Sets, III

Example: Level sets for $f(x, y) = x^2 + y^2$ on the 3D plot.



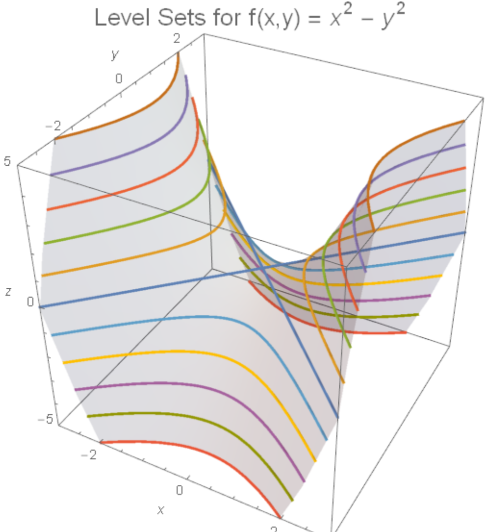
Level Sets, II

Example: The level sets for the function $f(x, y) = x^2 - y^2$ are hyperbolas in the plane.



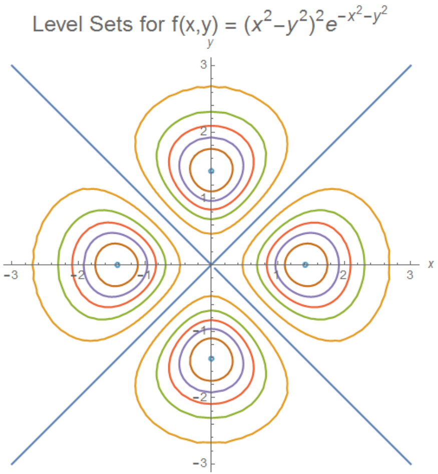
Level Sets, III

Example: Level sets for $f(x, y) = x^2 - y^2$ on the 3D plot.



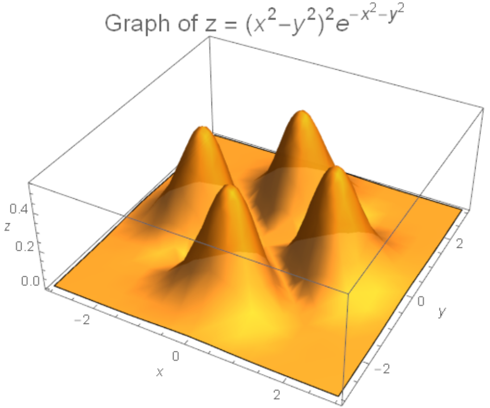
Level Sets, IV

Example: Level sets for the function $f(x, y) = (x^2 - y^2)^2 e^{-x^2-y^2}$.



Level Sets, V

Example: 3D plot of $f(x, y) = (x^2 - y^2)^2 e^{-x^2-y^2}$. Try picturing how the level sets arise from "horizontal slices" of this surface.



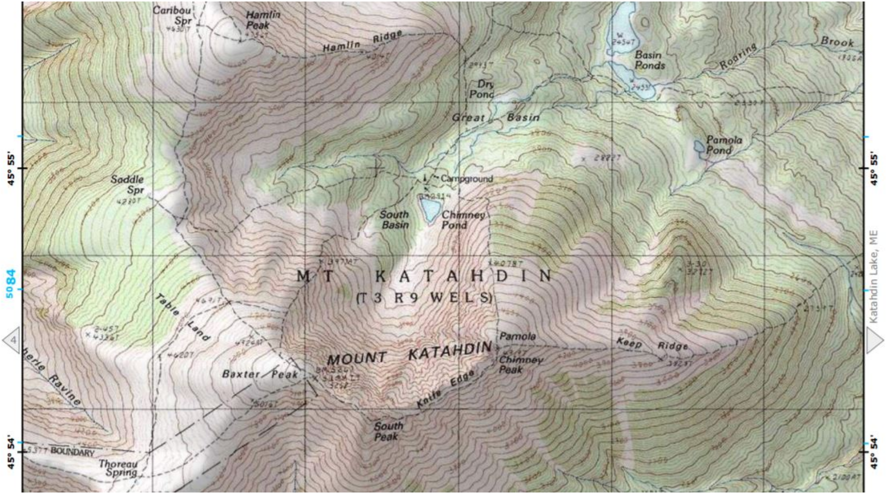
Level Sets, VI

We can read off the values of the function, and in particular identify where the values are changing from plots of the level curves.

- Many level curves bunched closely together indicates that the function is changing quickly so that the function's graph will be "steep".
- Inversely, having level curves grouped far apart means that the function's graph will be fairly flat.

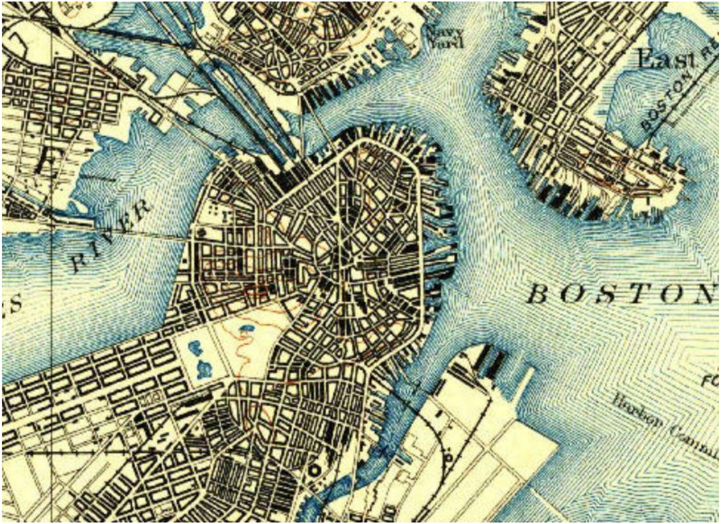
Level Sets, VII

Level curves give a topographical map of a function. Compare....



Level Sets, VIII

Level curves give a topographical map of a function. Compare....



Graphing Functions of 3 Variables, I

We can also represent some information graphically for functions $f(x, y, z)$ of three variables.

- Unfortunately, we cannot really produce proper graphs of these functions, since plotting a graph $w = f(x, y, z)$ would require drawing a 4-dimensional picture.
- However, we can still talk about level sets of functions of 3 variables: these are the points (x, y, z) satisfying a relation $f(x, y, z) = c$, which will (in general) give rise to <u>level surfaces</u> in 3-dimensional space.
- Note that the graphs $z = g(x, y)$ we have previously discussed are actually just special cases, since they are level surfaces for the function $f(x, y, z) = g(x, y) - z$. (Specifically, the points (x, y, z) with $z = g(x, y)$ are the level surface $f(x, y, z) = 0$).

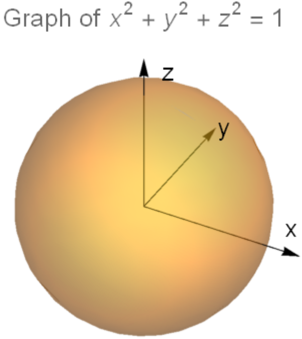
Graphing Functions of 3 Variables, II

Example: The level surfaces for the function $f(x, y, z) = x^2 + y^2 + z^2$ are spheres centered at the origin.

- Specifically, by the distance formula, the points satisfying $x^2 + y^2 + z^2 = c$ (for $c > 0$) are precisely those which are at a distance of $\sqrt{c}$ from the origin, which is just the sphere of radius $\sqrt{c}$ centered at $(0, 0, 0)$.

Graphing Functions of 3 Variables, II

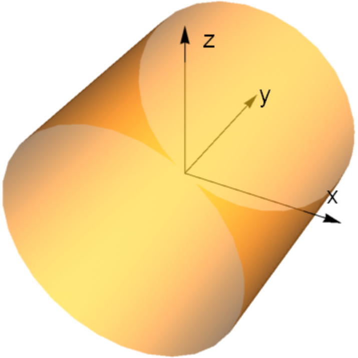
Example: Graph of the unit sphere $x^2 + y^2 + z^2 = 1$:



Graphing Functions of 3 Variables, III

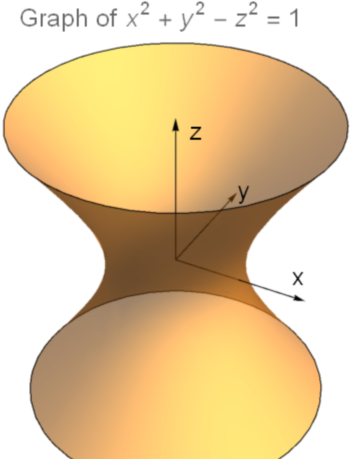
Example: The level surfaces for the function $f(x, y, z) = x^2 + z^2$ are cylinders whose axis is given by the y-axis:

Graph of $x^2 + z^2 = 1$



Graphing Functions of 3 Variables, IV

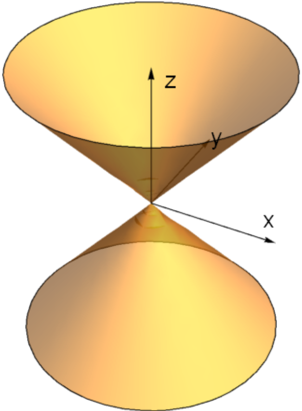
Example: The level surfaces for $f(x, y, z) = x^2 + y^2 - z^2$ yield different surfaces depending on the value of the function. For positive values, the shape is a hyperboloid of one sheet:



Graphing Functions of 3 Variables, V

Example: The level surfaces for $f(x, y, z) = x^2 + y^2 - z^2$ yield different surfaces depending on the value of the function. For the value 0, the shape is a double cone:

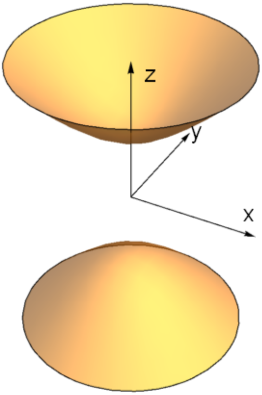
Graph of $x^2 + y^2 - z^2 = 0$



Graphing Functions of 3 Variables, VI

Example: The level surfaces for $f(x, y, z) = x^2 + y^2 - z^2$ yield different surfaces depending on the value of the function. For negative values, the shape is a hyperboloid of two sheets:

Graph of $x^2 + y^2 - z^2 = -1$



Quadric Surfaces, I

Many of the surfaces I have plotted today are examples of quadric surfaces: level sets of quadratic polynomials in x, y, z.

- Quadric surfaces are the 3-dimensional analogue of the famous conic sections (ellipse, parabola, hyperbola) often studied in precalculus courses.
- The general equation for a quadric surface is $Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gx + Hy + Iz = J$, for constants $A, B, C, D, E, F, G, H, I, J$.
- By changing variables, one can eliminate the cross terms, and then complete the squares to eliminate other linear terms.

Quadric Surfaces, II

One can eventually show that there are nine interesting types of quadric surfaces:

1. The ellipsoid (a "stretched sphere").
2. The hyperboloid of one sheet.
3. The hyperboloid of two sheets.
4. The paraboloid (our "parabolic bowl").
5. The hyperbolic paraboloid (our "saddle surface").
6. The elliptical cone (a "stretched double-cone").

There are also three families of cylinders:

7. The elliptical cylinder.
8. The parabolic cylinder.
9. The hyperbolic cylinder.

Quadric Surfaces, III

Many of the quadric surfaces have interesting or useful physical properties, and thus show up in engineering and industrial settings.

- The paraboloid is used as the shape of a satellite dish, since parallel rays coming from a long distance will all reflect into its focus.
- The hyperboloid of one sheet and the hyperbolic paraboloid are called ruled surfaces: through each point on the surface pass two lines which are contained in the surface.
- Ruled surfaces can therefore be (physically) constructed using materials which are not curved. The hyperboloid of one sheet, in particular, is a common design for cooling towers, while the saddle surface has been used as the roof design for certain sports arenas.

Quadric Surfaces, IV

Here is a picture of a cooling tower (on the right):

[Image credit: Wikipedia.]

Quadric Surfaces, V

Here is a picture of Calgary's Scotiabank Saddledome:

[Image credit: Ed Middleton/CBC.]

Summary

We discussed course logistics for Math 2321.

We discussed 3D coordinates, graphing, and level sets.

Next lecture: Vectors, dot and cross products.