Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													
out of	8	9	8	9	8	8	8	8	9	9	8	8	100

MATH 2321 Final Exam

December 17, 2020

Solutions

Instructor's name____________________ Your name____________________

Please check that you have 12 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1. Consider the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$.

(a) (3 points) Find the gradient ∇f of the function f.

$$\begin{aligned}\nabla f &= (f_x, f_y, f_z)\\ &= \left(3x^2\sqrt{y^2+z^2}, \frac{x^3 y}{\sqrt{y^2+z^2}}, \frac{x^3 z}{\sqrt{y^2+z^2}}\right)\end{aligned}$$

1 pt. for each component

(b) (3 points) Find the linearization of the function f at the point (2,3,4).

$$\begin{aligned}L_f(x, y, z; 2, 3, 4) &= f(2,3,4) + \nabla f(2,3,4) \cdot (x-2, y-3, z-4) \quad \text{1 pt.}\\ &= 40 + \left(60, \frac{24}{5}, \frac{32}{5}\right) \cdot (x-2, y-3, z-4)\\ &= 40 + 60(x-2) + 4.8(y-3) + 6.4(z-4). \quad \text{2 pts.}\end{aligned}$$

(c) (2 points) Use the linearization of f from part (b) to estimate the number $(1.98)^3\sqrt{(3.02)^2 + (4.01)^2}$.

$$\begin{aligned}(1.98)^3\sqrt{(3.02)^2+(4.01)^2} &\cong 40 + 60(1.98-2) + 4.8(3.02-3) + 6.4(4.01-4) \quad \text{1 pt.}\\ &= 40 + 60(-0.02) + 4.8(0.02) + 6.4(0.01)\\ &= 38.96. \quad \text{1 pt.}\end{aligned}$$

2. Let $f(x,y,z) = \ln(x^2 + 2y^2 + yz)$.

(a) (3 points) Find the rate of change of f with respect to z at the point $(1,1,2)$.

$$\frac{\partial f}{\partial z} = \frac{y}{x^2 + 2y^2 + yz} \quad \text{2 pts.}$$
$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,2)} = \frac{1}{5} \quad \text{1 pt.}$$

(b) (3 points) Find the minimum rate of change of f at the point $(1,1,2)$ **and** the direction in which this minimum rate of change occurs.

$$\nabla f(x,y,z) = \left(\frac{2x}{x^2 + 2y^2 + yz}, \frac{4y+z}{x^2 + 2y^2 + yz}, \frac{y}{x^2 + 2y^2 + yz}\right) \quad \text{1 pt.}$$
$$\nabla f(1,1,2) = \frac{1}{5}(2,6,1)$$
$$||\nabla f(1,1,2)|| = \frac{\sqrt{41}}{5}$$

The minimum rate of change is $-\frac{\sqrt{41}}{5}$. 1 pt.
The direction is $-\frac{1}{\sqrt{41}}(2,6,1)$. 1 pt.

(c) (3 points) Find a direction in which the value of f is not changing at the point $(1,1,2)$.

$$D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = 0 \quad \text{1 pt.}$$
$$\iff 0 = \frac{1}{5}(2,6,1) \cdot \boldsymbol{u} \quad \text{1 pt.}$$
$$\iff 0 = (2,6,1) \cdot \boldsymbol{u}.$$

Any unitary vector that satisfy the last equality is a valid answer. For example,

$$\boldsymbol{u} = \frac{1}{\sqrt{10}}(-3,1,0). \quad \text{1 pt.}$$

3. Consider the real valued function $F(x, y) = xe^{x-3} - y^3$.

(a) (4 points) Find an equation of the tangent line to the **level curve** $F(x, y) = 2$ at the point $(3, 1)$.

$$\begin{aligned} \nabla F(x,y) &= \left(e^{x-3} + xe^{x-3}, -3y^2\right) \quad \text{1 pt.} \\ \nabla F(3,1) &= \left(e^0 + (0)e^0, -3(1^2)\right) \\ &= (4, -3). \quad \text{1 pt.} \end{aligned}$$

$$\begin{aligned} (4,-3)\cdot(x-3, y-1) &= 0 \quad \text{1 pt.} \\ 4(x-3) - 3(y-1) &= 0. \quad \text{1 pt.} \end{aligned}$$

(b) (4 points) Find an equation of the tangent plane to the **graph of the function** F, with equation $z = F(x, y)$, at the point $(3, 1, 2)$.

$$\begin{aligned} L_f(x,y;3,1) &= F(3,1) + \nabla F(3,1)\cdot(x-3, y-1) \quad \text{1 pt.} \\ &= 2 + 4(x-3) - 3(y-1) \quad \text{1 pt.} \end{aligned}$$

An equation of the tangent plane is $z = 2 + 4(x-3) - 3(y-1)$. 2 pts.

4. Consider the function $f(x, y) = -3x^2 + 3xy - y^3$.

(a) (3 points) This function has **two** critical points. Find them both.

$$\begin{cases} f_x &= -6x + 3y \\ f_y &= 3x - 3y^2 \end{cases} \implies \begin{cases} -6x + 3y &= 0 \\ 3x - 3y^2 &= 0 \end{cases} \implies \begin{cases} y &= 2x \\ x &= y^2 \end{cases} \quad \text{1 pt.}$$

$$\begin{aligned} &x = 4x^2 \implies \\ &x = 0 \text{ or } x = \frac{1}{4} \\ &x = 0 \implies y = 0 \\ &x = \frac{1}{4} \implies y = \frac{1}{2}. \end{aligned}$$

The critical points are $(0, 0)$ and $\left(\frac{1}{4}, \frac{1}{2}\right)$. 2 pts.

(b) (6 points) Classify **both** critical points of the function f as local maximum, local minimum, or saddle points.

$$D(x, y) = \begin{vmatrix} -6 & 3 \\ 3 & -6y \end{vmatrix} \quad \text{2 pts.}$$

$$D(0, 0) = \begin{vmatrix} -6 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0 \quad \text{1 pt.}$$

$$\implies (0, 0) \quad \text{is a saddle point} \quad \text{1 pt.}$$

$$D\left(\frac{1}{4}, \frac{1}{2}\right) = \begin{vmatrix} -6 & 3 \\ 3 & -3 \end{vmatrix} = (-6)(-3) - 9 > 0 \text{ and } f_{xx}\left(\frac{1}{4}, \frac{1}{2}\right) < 0 \quad \text{1 pt.}$$

$$\implies \left(\frac{1}{4}, \frac{1}{2}\right) \quad \text{is a local maximum point.} \quad \text{1 pt.}$$

Note: If one point is incorrect, but the conclusion using that point is correct, then 1 pt. is given instead of the maximum of 3 pts.

5. (8 points) Using the **Lagrange Multiplier Method**, find the global minimum of the function

$$f(x, y, z) = x^2 + 3y^2 + z^2$$

on the plane with equation

$$x + 4y + 5z = 6.$$

No credit is given for using a different method than the Lagrange Multiplier method.

For $g(x, y, z) = x + 4y + 5z$, we set the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 6 \end{cases} \implies \begin{cases} 2x & = \lambda \\ 6y & = 4\lambda \\ 2z & = 5\lambda \\ x + 4y + 5z & = 6 \end{cases}$$ 4 pts.

$$\begin{cases} x & = \frac{1}{2}\lambda \\ y & = \frac{2}{3}\lambda \\ z & = \frac{5}{2}\lambda \end{cases} \implies \frac{1}{2}\lambda + \frac{8}{3}\lambda + \frac{25}{2}\lambda = 6 \implies \lambda = \frac{18}{47}.$$ 1 pt.

The critical point with respect to the constraint is $\left(\frac{9}{47}, \frac{12}{47}, \frac{45}{47}\right)$. 2 pts.

The minimum value is $f\left(\frac{9}{47}, \frac{12}{47}, \frac{45}{47}\right) = \frac{2538}{2209} = 1.15$. 1 pt.

6. Consider the plane region R bounded above by $y = 12 - x$ and $y = 5x$ and below by $y = x^2$.

(a) (2 points) Sketch the region R by labeling carefully all the boundaries.

The intersection points are $(2, 10)$, $(3, 9)$ and $(0, 0)$.
Correct region, with correct boundaries, not necessarily at scale: 1 pt.
All 3 boundaries clearly labeled: 1 pt.

(b) (6 points) Integrate $f(x, y) = 2x$ over the region R.

Method I:

$$\begin{aligned}\iint_R 2x\ dA &= \int_0^2 \int_{x^2}^{5x} 2x\ dy\ dx + \int_2^3 \int_{x^2}^{12-x} 2x\ dy\ dx \quad \text{2 pts. each integral}\\ &= \int_0^2 10x^2 - 2x^3\ dx + \int_2^3 24x - 2x^2 - 2x^3\ dx \quad \text{1 pt.}\\ &= \left(\frac{10x^3}{3} - \frac{x^4}{2}\right)\Big|_0^2 + \left(12x^2 - \frac{2x^3}{3} - \frac{x^4}{2}\right)\Big|_2^3\\ &= \frac{80}{3} - \frac{16}{2} + 12\cdot 9 - \frac{2\cdot 27}{3} - \frac{81}{2} - 12\cdot 4 + \frac{16}{3} + \frac{16}{2}\\ &= 33.5. \quad \text{1 pt.}\end{aligned}$$

Method II:

$$\begin{aligned}\iint_R 2x\ dA &= \int_0^9 \int_{\frac{y}{5}}^{\sqrt{y}} 2x\ dx\ dy + \int_9^{10} \int_{\frac{y}{5}}^{12-y} 2x\ dx\ dy \quad \text{2 pts. each integral}\\ &= \int_0^9 x^2\Big|_{\frac{y}{5}}^{\sqrt{y}} dy + \int_9^{10} x^2\Big|_{\frac{y}{5}}^{12-y} dy \quad \text{1 pt.}\\ &= \int_0^9 y - \frac{y^2}{25} dy + \int_9^{10} (12-y)^2 - \frac{y^2}{25} dy\\ &= \left(\frac{y^2}{2} - \frac{y^3}{75}\right)\Big|_0^9 + \left(-\frac{(12-y)^3}{3} - \frac{y^3}{75}\right)\Big|_9^{10}\\ &= \frac{81}{2} - \frac{9^3}{75} - \frac{8}{3} - \frac{1000}{75} + 9 + \frac{9^3}{75}\\ &= 33.5. \quad \text{1 pt.}\end{aligned}$$

7. (8 points) Evaluate the integral

$$\int_0^1 \int_x^{\sqrt{2-x^2}} y\,dydx,$$

by converting it to **polar coordinates**.
(Note: No points will be given, if polar coordinates are NOT used.)

$$R\colon \begin{cases} 0 \le & r \le \sqrt{2} \\ \frac{\pi}{4} \le & \theta \le \frac{\pi}{2} \end{cases} \qquad \text{3 pts.}$$

$$\begin{aligned}
\int_0^1 \int_x^{\sqrt{2-x^2}} y\,dy\,dx &= \int_0^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r\sin\theta \cdot r\,d\theta\,dr \quad \text{2 pts. for the function to be integrated} \\
&= \int_0^{\sqrt{2}} r^2 \left(-\cos\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right) dr \quad \text{1 pt.} \\
&= \frac{\sqrt{2}}{2} \int_0^{\sqrt{2}} r^2\,dr \\
&= \frac{\sqrt{2}}{2} \cdot \left(\frac{r^3}{3} \Big|_0^{\sqrt{2}} \right) \quad \text{1 pt.} \\
&= \frac{\sqrt{2}}{2} \cdot \frac{2\sqrt{2}}{3} \\
&= \frac{2}{3}. \quad \text{1 pt.}
\end{aligned}$$

Note: If r is forgotten at the change of variables, and the rest is correct, then the student will get 3 pts. instead of 5 pts. possible.

8. (8 points) Evaluate

$$\iiint_E z\, dV,$$

where E lies in the first octant $(x \geq 0,\ y \geq 0,\ z \geq 0)$ between the spheres with equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4.$$

$$E\colon \begin{cases} 1 \leq & \rho \leq 2 \\ 0 \leq & \theta \leq \frac{\pi}{2} \\ 0 \leq & \phi \leq \frac{\pi}{2} \end{cases} \qquad \text{3 pts. for the limits of integration}$$

$$\begin{aligned}
\iiint_E z\, dV &= \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho\cos\phi \cdot \rho^2 \sin\phi \; d\phi \; d\theta \; d\rho \quad \text{2 pts. for the function to be integrated} \\
&= \int_1^2 \int_0^{\frac{\pi}{2}} \rho^3 \cdot \frac{1}{2}(\sin\phi)^2 \Big|_0^{\frac{\pi}{2}} \; d\theta \; d\rho \quad \text{1 pt.} \\
&= \frac{\pi}{4} \int_1^2 \rho^3 \; d\rho \\
&= \frac{\pi}{4} \cdot \frac{\rho^4}{4} \Big|_1^2 \quad \text{1 pt.} \\
&= \frac{\pi}{16}(16 - 1) \\
&= \frac{15\pi}{16} \quad \text{1 pt.}
\end{aligned}$$

Note: If $\rho^2 \sin\phi$ is forgotten at the change of variables, and the rest is correct, then the student will get 3 pts. instead of 5 pts. possible.

9. (9 points) Find the mass of the solid under the paraboloid $z = x^2 + y^2$ and above the disc in the xy-plane where $x^2 + y^2 \le 3$, with the density

$$\delta(x, y, z) = 1 + x^2 + y^2 \text{ kg/m}^3.$$

Your answer should include **units**.

$$E: \begin{cases} 0 \le & r \le \sqrt{3} \\ 0 \le & \theta \le 2\pi \\ 0 \le & z \le r^2 \end{cases} \quad \text{3 pts. for the limits of integration}$$

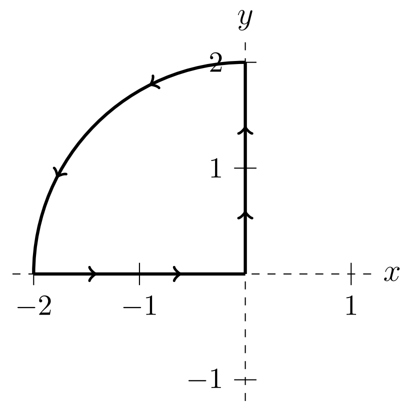
$$\begin{aligned}
\text{Mass} &= \iiint_E 1 + x^2 + y^2 \, dV \quad \text{1 pt.} \\
&= \int_0^{\sqrt{3}} \int_0^{2\pi} \int_0^{r^2} (1 + r^2) \cdot r \, dz \, d\theta \, dr \quad \text{2 pts. for the function to be integrated} \\
&= 2\pi \int_0^{\sqrt{3}} r^3 + r^5 \, dr \\
&= 2\pi \left(\frac{r^4}{4} + \frac{r^6}{6} \right) \Big|_0^{\sqrt{3}} \quad \text{1 pt.} \\
&= 2\pi \left(\frac{9}{4} + \frac{27}{6} \right) \\
&= \frac{27\pi}{2} \text{kg} \quad \text{1 pt. for answer and 1 pt. for unit}
\end{aligned}$$

10. (9 points)

Calculate the line integral of

$$\boldsymbol{F}(x, y) = (x - y^3, x^3 + y^5)$$

along the curve $C = C_1 + C_2 + C_3$, where
C_1 is the line segment from $(0, 0)$ to $(0, 2)$,
C_2 is a quarter-circle from $(0, 2)$ to $(-2, 0)$,
C_3 is a line segment from $(-2, 0)$ to $(0, 0)$, as shown in the picture to the right.



Method I:

2-curl($\boldsymbol{F}$) $= Q_x - P_y = 3x^2 - (-3y^2) = 3(x^2 + y^2)$.

If R denotes the region bounded by C, then by Green's Theorem 1 pt. we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = + \iint_R 3(x^2 + y^2)\ dA \quad \text{2 pts.}$$

$$= \int_0^2 \int_{\frac{\pi}{2}}^{\pi} 3r^2 \cdot r\ dr\ d\theta \quad \text{3 pts. for limits and 2 pts. for function}$$

$$= \frac{\pi}{2} \int_0^2 3r^3\ dr = \frac{3\pi}{8} r^4 \Big|_0^2$$

$$= 6\pi. \quad \text{1 pt.}$$

Method II:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} \quad \text{1 pt.}$$

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^2 (*, t^5) \cdot (0, 1)\ dt = \int_0^2 t^5\ dt \quad \text{1 pt.}$$

$$= \frac{t^6}{6}\Big|_0^2 = \frac{32}{3}. \quad \text{1 pt. for answer}$$

$$\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\frac{\pi}{2}}^{\pi} (2\cos t - 8\sin^3 t, 8\cos^3 t + 32\sin^5 t) \cdot (-2\sin t, 2\cos t)\ dt \quad \text{1 pt.}$$

$$= 4\int_{\frac{\pi}{2}}^{\pi} -\cos t \sin t + 16\cos t \sin^5 t + 3 + \cos(4t)\ dt \quad \text{1 pt.}$$

$$= 4\left(\frac{1}{2}\cos^2 t + \frac{8}{3}\sin^6 t + 3t + \frac{1}{4}\sin(4t)\right)\Big|_{\frac{\pi}{2}}^{\pi} = 2 - \frac{32}{3} + 6\pi \quad \text{1 pt. for answer}$$

$$\int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{-2}^{0} (t, *) \cdot (1, 0)\ dt = \int_{-2}^{0} t\ dt \quad \text{1 pt.}$$

$$= \frac{t^2}{2}\Big|_{-2}^{0} = -2. \quad \text{1 pt. for answer}$$

By adding all three values above we obtain $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 6\pi.$ 1 pt.

11. (8 points) Evaluate the flux integral $\iint_M \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ where

$$\boldsymbol{F}(x, y, z) = \big(x, y, 3\big)$$

and M is the surface that consists of the cylinder

$$\{x^2 + y^2 = 4,\ 0 \le z \le 2\}$$

plus the disk

$$\{x^2 + y^2 \le 4,\ z = 0\}$$

on the bottom of this cylinder. The surface M is oriented by the outward normal.

Method I: Let M' be the disk: $\{x^2 + y^2 \le 4,\ z = 2\}$. 1 pt.

Since $M + M'$ is closed, we may apply the Divergence Theorem to calculate:

$$\begin{aligned}\iint_{M+M'} \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= +\iiint_E \nabla \cdot \boldsymbol{F}\ dV \quad \text{2 pts.}\\ &= \iiint_E 2\ dV \quad \text{1 pt.}\\ &= 2 \cdot \text{volume}(E) = 2 \cdot 2 \cdot \pi \cdot 2^2 = 16\pi. \quad \text{1 pt. for answer}\end{aligned}$$

We apply the definition to calculate:

$$\begin{aligned}\iint_{M'} \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iint_{M'} (x, y, 3) \cdot (0, 0, 1) dS \quad \text{1 pt.}\\ &= \iint_{M'} 3\ dS \quad \text{1 pt.}\\ &= 3 \cdot \text{area}(M') = 12\pi. \quad \text{1 pt. for answer}\end{aligned}$$

We conclude that $\iint_M \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_{M+M'} \boldsymbol{F} \cdot \boldsymbol{n}\, dS - \iint_{M'} \boldsymbol{F} \cdot \boldsymbol{n}\, dS = 4\pi.$ 1 pt. for answer

Method II: Let M_1 denote the cylinder and M_2 denote the disk, thus $M = M_1 + M_2$.

$$\begin{aligned}\iint_{M_1} \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \int_0^{2\pi} \int_0^2 (2\cos u, 2\sin u, 3) \cdot (2\cos u, 2\sin u, 0)\ dv\ du \quad \text{2 pts.}\\ &= \int_0^{2\pi} \int_0^2 4\cos^2 u + 4\sin^2 u\ dv\ du = \int_0^{2\pi} \int_0^2 4\ dv\ du \quad \text{1 pt.}\\ &= 4 \cdot 2\pi \cdot 2 = 16\pi. \quad \text{1 pt. for answer}\end{aligned}$$

$$\begin{aligned}\iint_{M_2} \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iint_{M_2} (x, y, 3) \cdot (0, 0, -1)\ dS \quad \text{1 pt.}\\ &= \iint_{M_2} -3\ dS \quad \text{1 pt.}\\ &= 3 \cdot \text{area}(M_2) = -12\pi. \quad \text{1 pt. for answer}\end{aligned}$$

We conclude that $\iint_M \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_{M_1} \boldsymbol{F} \cdot \boldsymbol{n}\, dS + \iint_{M_2} \boldsymbol{F} \cdot \boldsymbol{n}\, dS = 4\pi.$ 1 pt. for answer

12. Consider the vector field $\boldsymbol{F}(x,y,z) = \big(2, xz, z^3\big)$.

(a) (3 points) Calculate the curl of $\boldsymbol{F}$.

$$\operatorname{curl}(\boldsymbol{F}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ 2 & xz & z^3 \end{vmatrix} = (-x, 0, z). \quad \text{1 pt. for each component}$$

(b) (5 points) Using Stokes' Theorem evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the boundary of the surface $z = xy^2$, $0 \leq x, y \leq 1$, oriented counterclockwise as viewed from above.

Let M be the surface that is bounded by the curve C.
The parametrization of M is $\boldsymbol{r}(x,y) = (x, y, xy^2)$ over D that is the square $0 \leq x, y \leq 1$.
The orientation of M is upward, positive z component.
$\boldsymbol{r}_x \times \boldsymbol{r}_y = (-y^2, -2xy, 1)$.
By Stokes'Theorem

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot \, d\boldsymbol{r} &= \iint_M \operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n} \, dS \quad \text{1 pt.} \\
&= \iint_D (-x, 0, x^2 y) \cdot (-y^2, -2xy, 1) \, dA \quad \text{2 pts.} \\
&= \int_0^1 \int_0^1 2xy^2 \, dx \, dy \quad \text{1 pt.} \\
&= 2 \left(\int_0^1 x \, dx \right) \cdot \left(\int_0^1 y^2 \, dy \right) \\
&= 2 \cdot \frac{1}{2} \cdot \frac{1}{3} \\
&= \frac{1}{3}. \quad \text{1 pt.}
\end{aligned}$$