Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													
out of	9	9	9	9	9	9	9	9	7	7	7	7	100

Math 2321 Final Exam

December 12, 2017

Instructor's name____________________ Your name____________________

Please check that you have 12 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1) (9 points) Suppose that the xy-plane is occupied by a charged metal plate. Let $E = E(x, y)$ be the electric charge, in coulombs, of the plate at the point (x, y) meters. Suppose that

$$\left.\frac{\partial E}{\partial x}\right|_{(1,2)} = 3 \text{ C/m} \quad \text{and} \quad \left.\frac{\partial E}{\partial y}\right|_{(1,2)} = -2 \text{ C/m}.$$

A particle is moving along the plate so that it position, at time t seconds, is $(x, y) = (t^2, t^3 + t)$. Find the rate of change of the electric charge at the particle's location, in C/s, at time $t = 1$ second.

$(x,y)\big|_{t=1} = (1,2).$ $\quad x'(t) = 2t.$ $\quad y'(t) = 3t^2+1.$

$$\left.\frac{dE}{dt}\right|_{t=1} = \left.\frac{\partial E}{\partial x}\right|_{(1,2)} \cdot \left.\frac{dx}{dt}\right|_{t=1} + \left.\frac{\partial E}{\partial y}\right|_{(1,2)} \cdot \left.\frac{dy}{dt}\right|_{t=1}$$

$$= (3)(2) + (-2)(4) = \boxed{-2 \text{ C/s}.}$$

2) (9 points) The air temperature in a room at a point (x, y, z) is given by

$$T(x, y, z) = z(x-1)^2 + y^3 \ \ {}^\circ\mathrm{C},$$

where x, y, z are measured in meters. In which direction (as a unit vector in $\mathbb{R}^3$) should a bug at the point $(2, 1, 1)$ fly in order to cool off the fastest, and at what rate (in degrees Celsius per meter) will the air temperature around it change if it moves in that direction?

$$\vec{\nabla} T = \left(2z(x-1),\ 3y^2,\ (x-1)^2\right).$$

$$\vec{\nabla} T(2,1,1) = \left(2\cdot 1\cdot(2-1),\ 3(1)^2,\ (2-1)^2\right)$$

$$= (2,3,1).$$

In direction of

$$\frac{-\vec{\nabla} T(2,1,1)}{|\vec{\nabla} T(2,1,1)|} = \frac{-(2,3,1)}{\sqrt{14}}.$$

Rate in that direction:

$$-\sqrt{14}\ {}^\circ\mathrm{C}/\mathrm{m}.$$

3) (9 points) Consider the surface M given by $\underbrace{(x-1)^2+y^3-z}=0$. At which points on M is the tangent plane parallel to the plane given by $\underbrace{4x-27y+z}=2$?

Non-zero orthog. vector: $(4,-27,1)$.

$F(x,y,z)$

~~$\vec{\nabla} F$~~

$$\vec{\nabla}F(x,y,z) = (2(x-1), 3y^2, -1).$$

Want:

$$(x-1)^2+y^3-z=0 \quad \text{and}$$

$$(2(x-1), 3y^2, -1) = \lambda(4,-27,1).$$

$$2(x-1)=4\lambda. \quad 3y^2=-27\lambda. \quad -1=\lambda.$$

$$2(x-1)=-4. \quad 3y^2=27.$$

$$x-1=-2. \quad y^2=9,$$

$$\boxed{x=-1} \quad \boxed{y=\pm 3.}$$

$x=-1.$

$y=-3,$

$$z=(x-1)^2+y^3$$
$$=(-2)^2+(-3)^3$$
$$=4-27=-23.$$

$x=-1,$

$y=3,$

$$z=(x-1)^2+y^3$$
$$=(-2)^2+3^3$$
$$=4+27=31,$$

$$\boxed{(x,y,z)=(-1,-3,-23) \text{ ~~and~~ or } (-1,3,31).}$$

4) (9 points) Consider the surface M given by the parametrization $\mathbf{r}(u,v) = (u^3, v^3, uv)$. Give a parametric equation for the tangent plane to M at the point where $(u,v) = (1,-1)$. Also calculate a normal vector to M at this point.

$$\underline{r}_u = (3u^2, \cancel{3}0, v). \qquad \underline{r}(1,-1) = (1,-1,-1).$$

$$\underline{r}_v = (0, 3v^2, u).$$

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} i & j & k \\ 3u^2 & 0 & v \\ 0 & 3v^2 & u \end{vmatrix} = (-3v^3, -3u^3, 9u^2v^2).$$

$$\underline{r}_u(1,-1) = (3,0,-1), \qquad \underline{r}_u \times \underline{r}_v \Big|_{(1,-1)} = (3,-3,9).$$

$$\underline{r}_v(1,-1) = (0,3,1).$$

Parametric equation for tangent plane:

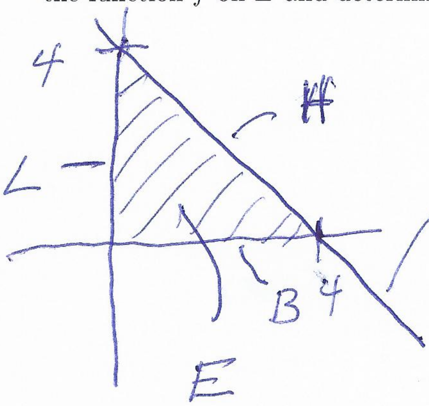
$$(x,y,z) = (1,-1,-1) + s(3,0,-1) + t(0,3,1).$$

Normal vector: $(1,-1,3)$ (or $(3,-3,9)$).

5) (9 points) Consider the function

$$f(x,y) = xy - x - y + 1$$

on the set E of all points (x,y) such that $x \geq 0$, $y \geq 0$ and $x + y \leq 4$. Find the global maximum and minimum values of the function f on E and determine the points at which these values are attained.



(x,y)	$f(x,y)$
$(1,1)$	0
$(0,0)$	1
$(0,4)$	-3
$(4,0)$	-3
$(2,2)$	1

Max. value: 1

Min. value: -3.

Interior crit. pts.:

$\frac{\partial f}{\partial x} = y - 1 = 0.$ $\boxed{(x,y) = (1,1).}$

$\frac{\partial f}{\partial y} = x - 1 = 0.$

On L: $x = 0$. $0 \leq y \leq 4$.

$f_L(y) = -y + 1$.

$f_L'(y) = -1 \neq 0$.

Endpts. $(0,0)$ and $(0,4)$

On B: $y = 0$. $0 \leq x \leq 4$,

$f_B(x) = -x + 1$,

$f_B'(x) = -1 \neq 0$.

Endpts. $(0,0)$ and $(4,0)$.

On H: $y = 4 - x$.
$0 \leq x \leq 4$.

$f_H(x) = x(4-x) - x$
$- (4 - x) + 1$.

$f_H(x) = 4x - x^2 - x$
$-4 + x + 1$

$= -x^2 + 4x - 3$.

$f_H'(x) = -2x + 4 = 0$,
$x = 2$. $y = 2$,

$\boxed{(2,2)}$

Endpts. $(0,4)$ and $(4,0)$

6) (9 points) Evaluate the iterated integral

$$\int_0^2 \int_{x\sqrt{3}}^{\sqrt{16-x^2}} \sin(x^2+y^2)\, dy\, dx\,.$$

$$= \int_{\pi/3}^{\pi/2} \int_0^4 (\sin(r^2))\, r\, dr\, d\theta =$$

$$\int_{\pi/3}^{\pi/2} \int_0^{16} \frac{1}{2} \sin u\, du\, d\theta$$

Let $u = r^2$,
$du = 2r\,dr$,
$\frac{1}{2} du = r\,dr$.

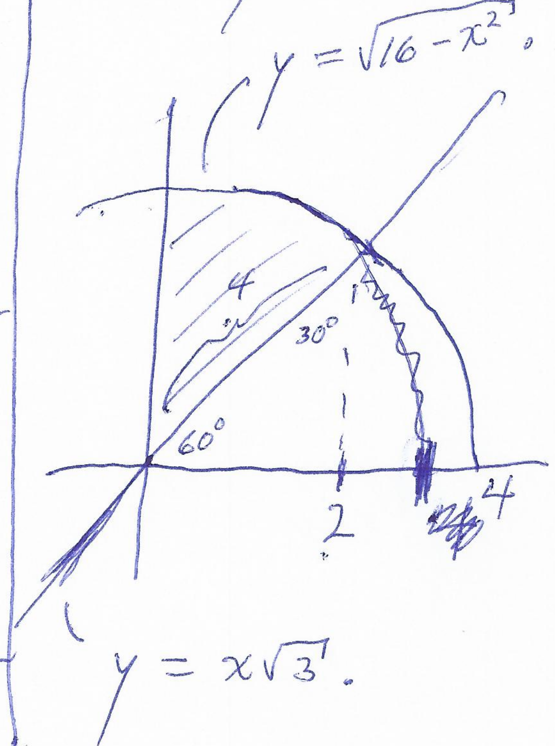
$$= \int_{\pi/3}^{\pi/2} -\frac{1}{2}\cos u \Big|_0^{16} d\theta$$

$$= \left(-\frac{1}{2}\cos(16) + \frac{1}{2}\right)\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \boxed{\frac{\pi}{12}(1 - \cos(16))}.$$

$0 \le x \le 2$.

$x\sqrt{3} \le y \le \sqrt{16-x^2}$.



$x\sqrt{3} = \sqrt{16-x^2}$

$3x^2 = 16 - x^2$.

$4x^2 = 16$,

$x^2 = 4$, $x = \pm 2$,

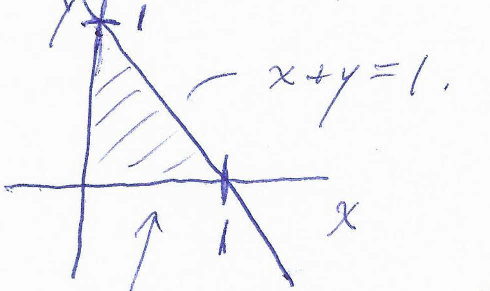
$0 \le r \le 4$.

$\pi/3 \le \theta \le \pi/2$,

7) (9 points) Consider the solid region T between the planes given by $x + 2y - z = 0$ and $3x + 4y - z + 2 = 0$ above the filled-in triangle in the xy-plane which has vertices given by $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 1, 0)$. Suppose that

$$f(x, y, z) = \frac{1}{1 + x + y}.$$

Calculate $\iiint_T f(x, y, z)\, dV$. (Hint: If you do this right, all of the integrals are very easy.)



Projected region R in xy-plane

$z = x + 2y$ and $z = 3x + 4y + 2.$

$$\iiint_T f\, dV =$$

$$\int_0^1 \int_0^{1-x} \int_{x+2y}^{3x+4y+2} \frac{1}{1+x+y}\, dz\, dy\, dx =$$

$$\int_0^1 \int_0^{1-x} \frac{1}{1+x+y}\left[3x+4y+2-(x+2y)\right] dy\, dx$$

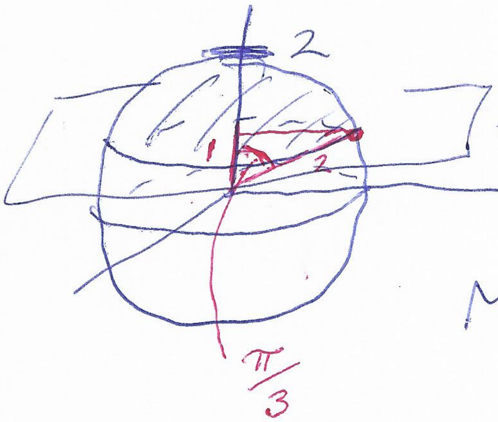
$$= \int_0^1 \int_0^{1-x} \frac{1}{1+x+y}(2x+2y+2)\, dy\, dx =$$

$$\iint_R 2\, dA = 2\left(\text{area of } R\right) = 2 \cdot \frac{1}{2}(1)(1) = \boxed{1.}$$

8) (9 points) Let T be the solid region which is inside the sphere given by $x^2 + y^2 + z^2 = 4$ and above the plane where $z = 1$. Assume that x, y, and z are measured in meters. Suppose the density of the solid region S is given by

$$\delta(x, y, z) = \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \text{ kg/m}^3.$$

Find the total mass of the solid region T.



$z = 1$, $\rho \cos\phi = 1$, $\rho = \frac{1}{\cos\phi}$.

$$\text{Mass} = \iiint_T \delta \, dV =$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\frac{1}{\cos\phi}}^{2} \frac{\cancel{\rho}\cos\phi}{\cancel{\rho^3}} \cancel{\rho^2}\sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}} \left(2 - \frac{1}{\cos\phi}\right)\cos\phi\sin\phi \, d\phi \, d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi/3} (2\cos\phi\sin\phi - \sin\phi) \, d\phi \, d\theta =$$

$$\int_0^{2\pi} \left(\sin^2\phi + \cos\phi\right)\Big|_0^{\pi/3} d\theta = \int_0^{2\pi}\left(\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} - 1\right) d\theta$$

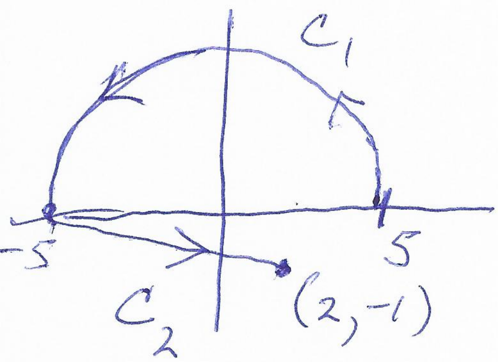
$$= \frac{1}{4} 2\pi = \frac{1}{2}\pi \text{ kg.}$$

9) Let $f(x,y) = xy^2 - x^2$.

a) (2 points) Let $\mathbf{F}(x,y) = \vec{\nabla} f(x,y)$. Calculate the vector field $\mathbf{F}(x,y)$.

$$\underline{F} = (y^2 - 2x, 2xy).$$

b) (5 points) Let $\mathbf{F} = \mathbf{F}(x,y)$ be the vector field from part a). Let $C = C_1 + C_2$ be the oriented curve where C_1 is the upper half of the circle of radius 5, center at the origin, oriented counterclockwise, and C_2 is the oriented line segment from $(-5,0)$ to $(2,-1)$. Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

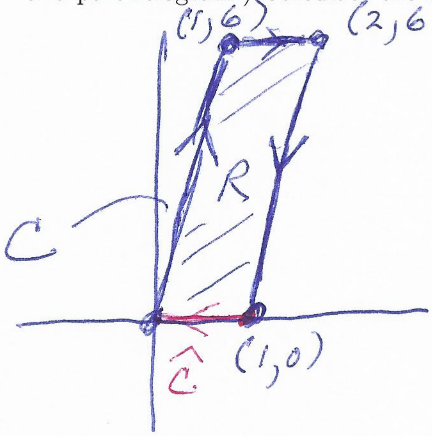


$$\int_C \underline{F} \cdot d\underline{r} = f(2,-1) - f(5,0)$$

$$= 2(-1)^2 - 2^2 - (0 - 25)$$

$$= -2 + 25 = \boxed{23.}$$

10) (7 points) Let $\mathbf{F}(x,y) = (\overbrace{1-y}^{P}, \overbrace{-2x+e^{y^2}}^{Q})$. Let C be the oriented curve which starts at $(0,0)$, goes along a straight line to $(1,6)$, then goes along a straight line to $(2,6)$, and finally goes along a straight line to $(1,0)$. (So that C is three sides of a parallelogram.) Calculate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.



$$\int_C F\cdot dr = \underbrace{\int_{C+\hat{C}} F\cdot dr}_{-\iint_R (Q_x - P_y)\,dA} - \int_{\hat{C}} F\cdot dr.$$

$$\int_C F\cdot dr = -\iint_R -2-(-1)\,dA - \int_{\hat{C}} \underbrace{P}_{1-y=1-0}\,dx + Q\,\underbrace{dy}_{0}$$

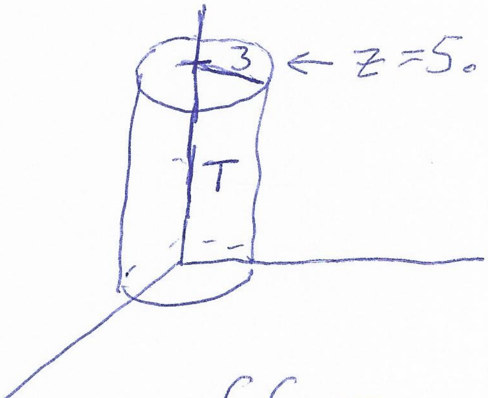
$$= \left(\text{Area of } R\right) - \int_1^0 1\,dx$$

$$= 1(6) - \left(x\big|_1^0\right) = 6-(0-1) \boxed{=7.}$$

11) (7 points) Let T be the solid right circular cylinder where $x^2 + y^2 \le 9$ and $0 \le z \le 5$. Let M be the boundary of T, oriented outward. Find the flux of the vector field

$$\mathbf{F}(x, y, z) = \left(7x + xy^2,\ \cos x + e^z,\ -zy^2\right)$$

through M.



Divergence Thm.

(2 points

$$\iint_M \underline{F} \cdot \underline{n}\, dS = \iiint_T (\vec{\nabla} \cdot \underline{F})\, dV$$

$$= \iiint_T \underbrace{7 + y^2 + 0 - y^2}_{2 \text{ points}}\, dV = \iiint_T 7\, dV$$

3 points

$$= 7(\text{volume of } T) = 7\pi(3)^2 5 = \boxed{315\pi.}$$

OR! or!

2 pts.

$$= \iiint_T 7\, dV = \int_0^{2\pi}\int_0^3\int_0^5 7r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 7rz\Big|_{z=0}^{z=5} dr\, d\theta = \int_0^{2\pi}\int_0^3 35r\, dr\, d\theta =$$

1 pt.

$$\int_0^{2\pi} \frac{35r^2}{2}\Big|_{r=0}^{r=3} d\theta = \int_0^{2\pi} \frac{315}{2}\, d\theta = \frac{315}{2}\theta\Big|_0^{2\pi} = \boxed{315\pi.}$$

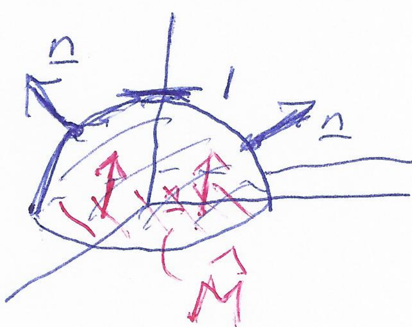
12) Let $\mathbf{F} = \left(e^{z^2} - y,\ e^{z^3} + x,\ \cos(xz)\right)$, and let M be the upper hemisphere of the sphere of radius 1, centered at the origin, with the upward orientation.

a) (2 points) Compute curl $\mathbf{F}$.

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{z^2} - y & e^{z^3} + x & \cos(xz) \end{vmatrix} =$$

$$\left(0 - 3z^2 e^{z^3},\ -\left(-z\sin(xz) - 2ze^{z^2}\right),\ \underbrace{1-(-1)}_{2}\right).$$

b) (5 points) Compute the flux of curl $\mathbf{F}$ through M.



by Stokes' Theorem

$$\iint_M (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS = \iint_{\hat{M}} (\vec{\nabla} \times \underline{F}) \cdot \underline{n}\, dS$$

$$= \iint_{\hat{M}} (*, *, 2) \cdot (0, 0, 1)\, dA =$$

$$2\left(\begin{matrix}\text{area} \\ \text{of } \hat{M}\end{matrix}\right) = 2\pi(1)^2 = \boxed{2\pi.}$$