Do not write in the boxes immediately below.

problem	1	2	3	4	5	6	7	8	9	10	11	12	total
points													

Math 2321 Final Exam

December 14, 2015

Instructor's name________________ Your name________________

Please check that you have 9 different pages.

Answers from your calculator, without supporting work, are worth zero points.

1) Consider the function $f(x,y) = 4x^4y^4 - \sqrt{x}\, e^{y-1}$.

a) (6 points) What is the linearization $L(x,y)$ of $f(x,y)$ at $\mathbf{a} = (1,1)$?

$f(1,1) = 4 - e^0 = 3.$

$f_x = 16x^3y^4 - \frac{1}{2}x^{-1/2}e^{y-1}. \quad f_x(1,1) = 16 - \frac{1}{2} = \frac{31}{2}.$

$f_y = 16x^4y^3 - \sqrt{x}\,e^{y-1}. \quad f_y(1,1) = 16 - 1 = 15.$

ANSWER: $L(x,y) = 3 + \frac{31}{2}(x-1) + 15(y-1).$

b) (2 points) Use your linearization to estimate $f(1.01, 0.99)$. (The "exact" answer without using linearization is worth 0 points.)

$f(1.01, 0.99) \approx L(1.01, 0.99)$

$= 3 + 15.5(0.01) + 15(-0.01)$

$= 3 + 0.155 - 0.15 = 3.005.$

ANSWER: 3.005

2) (8 points) Suppose that the real-valued function $f = f(x, y, z)$ is differentiable at $(-3, \ln 5, 1)$, and that $\nabla f(-3, \ln 5, 1) = (-3, 5, 2)$. Suppose that the position of a particle at time t is given by

$$x(t) = -3e^t, \quad y(t) = \ln(t^2 + 5), \quad z(t) = \cos t.$$

Calculate df/dt at $t = 0$.

$p(t) = (-3e^t, \ln(t^2+5), \cos t)$.

$p'(t) = (-3e^t, \frac{2t}{t^2+5}, -\sin t)$.

$p'(0) = (-3, 0, 0)$. $p(0) = (-3, \ln 5, 1)$.

$\frac{df}{dt}\Big|_{t=0} = \vec{\nabla} f(p(0)) \cdot p'(0) = (-3, 5, 2) \cdot (-3, 0, 0) = 9$.

ANSWER: 9

3) (4 points each) The elevation above sea level, in meters, of a saddle between two mountains is given by $z = xy + 1000$, where x and y are also in meters. Suppose that a goat is at a point near the saddle where $(x, y) = (10, -20)$.

a) In what direction (as a unit vector in the xy-plane) should the goat head to ascend as rapidly as possible?

$\vec{\nabla} z = (y, x)$. $\vec{\nabla} z(10, -20) = (-20, 10) = 10(-2, 1)$.

ANSWER: $\frac{(-2, 1)}{\sqrt{5}}$

b) At what rate, in meters of elevation per meter of distance in the xy-plane, will the goat ascend if it moves in the direction from (a)?

$|\vec{\nabla} z(10, -20)| = |10(-2, 1)| = 10\sqrt{5}$.

ANSWER: $10\sqrt{5}$.

4) (8 points) Find an equation for the tangent plane of the level surface of the function $f(x, y, z) = x^2 y - \sin z$ at the point $\mathrm{p} = (2, 1, 0)$.

$\vec{\nabla} f = (2xy, x^2, -\cos z)$.

$\vec{\nabla} f(2, 1, 0) = (4, 4, -1)$.

ANSWER: $4(x-2) + 4(y-1) - (z-0) = 0$.

5) (8 points) Consider the surface parameterized by $\mathbf{r}(u,v) = \left(u - v, u + v, \frac{u^2 - v^2}{2}\right)$. Describe, in parametric form, the tangent plane of $\mathbf{r}$ at $(u,v) = (4,2)$.

$\underline{r}_u = (1, 1, u)$. $\underline{r}_u(4,2) = (1,1,4)$.

$\underline{r}_v = (-1, 1, -v)$. $\underline{r}_v(4,2) = (-1,1,-2)$.

$\underline{r}(4,2) = (2, 6, \frac{16-4}{2}) = (2,6,6)$.

ANSWER: $(x,y,z) = (2,6,6) + a(1,1,4) + b(-1,1,-2)$.

6) (8 points) Find and classify the critical points of the function $f(x,y) = x^3 - 6xy + 8y^3$ as points where f attains a local maximum value, a local minimum value, or has a saddle point.

$f_x = 3x^2 - 6y = 0$. $x^2 = 2y$.

$f_y = -6x + 24y^2 = 0$. $-x + 4y^2 = 0$. $4y^2 = x$.

$(4y^2)^2 = 2y$. $16y^4 = 2y$. $y = 0$ or $8y^3 = 1$.

$y = 0$ or $y = (\frac{1}{8})^{1/3} = \frac{1}{2}$. $x = 4y^2$.

Crit. pts.: $(x,y) = (0,0), (1, \frac{1}{2})$.

$f_{xx} = 6x$. $f_{yy} = 48y$. $f_{xy} = -6$.

$D = \begin{vmatrix} 6x & -6 \\ -6 & 48y \end{vmatrix} = 288xy - 36$.

At $(0,0)$, ~~$D<0$~~ $D = -36 < 0$.

At $(1, \frac{1}{2})$, $D = 144 - 36 > 0$. $f_{xx} = 6 > 0$.

ANSWER: Saddle point at $(0,0)$.

Local minimum value at $(1, \frac{1}{2})$.

7) (10 points) Find the global maximum and global minimum values of $f(x, y) = x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 = 9$. ← Defines a circle, which is compact.

$g(x,y)$

$\vec{\nabla} f = \lambda \vec{\nabla} g$. $(2x, 4y-4) = \lambda(2x, 2y)$.

$2x = \lambda(2x)$. ⟶ $x = 0$ or $\lambda = 1$.

$4y - 4 = \lambda(2y)$.

$x^2 + y^2 = 9$.

If $x = 0$, then $y^2 = 9$; so $y = \pm 3$.

If $\lambda = 1$, $4y - 4 = 2y$. So, $2y = 4$. $y = 2$. $x^2 + 4 = 9$. $x^2 = 5$. $x = \pm\sqrt{5}$.

4 critical points for f subject to constraint:

$(0,-3)$, $(0,3)$, $(-\sqrt{5}, 2)$, $(\sqrt{5}, 2)$

(x,y)	$x^2 + 2y^2 - 4y$
$(0,-3)$	30 ← max.
$(0,3)$	6
$(-\sqrt{5}, 2)$	$5 + 2\cdot 4 - 4\cdot 2 = 5$ ← min.
$(\sqrt{5}, 2)$	$5 + 2\cdot 4 - 4\cdot 2 = 5$ ← min.

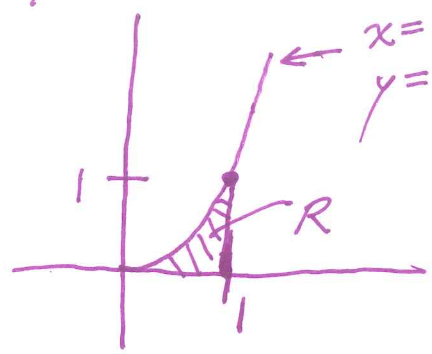
ANSWER: Maximum value = 30.

Minimum value = 5.

8) (8 points) Evaluate the iterated integral $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\, dx\, dy$.

$0 \le y \le 1.$

$\sqrt{y} \le x \le 1.$



Reverse order of integration.

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\, dx\, dy$$

$$= \iint_R (2+x^3)^{1/2}\, dA =$$

$$\int_0^1 \int_0^{x^2} (2+x^3)^{1/2}\, dy\, dx = \int_0^1 (2+x^3)^{1/2}\, y \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 (2+x^3)^{1/2} x^2\, dx = \int_2^3 u^{1/2} \left(\tfrac{1}{3}\, du\right)$$

Let $u = 2 + x^3$.
$du = 3x^2\, dx$.
$\frac{1}{3} du = x^2\, dx$.

$$= \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} \Big|_2^3 =$$

$$\tfrac{2}{9}\left(3^{3/2} - 2^{3/2}\right).$$

ANSWER: $\tfrac{2}{9}\left(3^{3/2} - 2^{3/2}\right)$

9) (8 points) An ice cream shop sells ice cream at a price of 2 dollars per pound. For one serving of ice cream, the machine fills the cone, which has a shape given by $z = \sqrt{3x^2 + 3y^2}$, and then continues adding ice cream up to the surface of the sphere given by $x^2 + y^2 + z^2 = 25$, where x, y, and z are in inches. Suppose that the ice cream they serve has a density of 0.04 lb/in^3. How much will it cost to get one serving of ice cream?

$z = \sqrt{3}\sqrt{x^2+y^2} = \sqrt{3}\, r.$

$\rho\cos\phi = \sqrt{3}\,\rho\sin\phi.$

$\tan\phi = \frac{1}{\sqrt{3}}. \quad \phi = \frac{\pi}{6}.$

$x^2+y^2+z^2=25,$

$\rho^2 = 25.$

$\rho = 5.$

$\text{Cost} = \left(\$2/\text{lb}\right)\left(0.04\,\frac{\text{lb}}{\text{in}^3}\right)\left(\text{volume in in}^3\right).$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^5 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \left(\frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=5}\right) d\phi\, d\theta =$$

$$\frac{125}{3}\int_0^{2\pi}\int_0^{\pi/6} \sin\phi \, d\phi\, d\theta =$$

$$\frac{250\pi}{3}\left(-\cos\phi\right)\Big|_0^{\pi/6} = \frac{250\pi}{3}\left(-\frac{\sqrt{3}}{2}+1\right).$$

$$\underset{\text{in }\$}{\text{Cost}} = (0.08)\left(\frac{250\pi}{3}\right)\left(1-\frac{\sqrt{3}}{2}\right) \approx 2.81$$

ANSWER: \$ 2.81

10) Suppose $\mathbf{F}(x, y) = (y^3e^x - x, 3y^2e^x + 4x)$ represents a force field in Newtons, where x, y, and z are in meters.

P Q

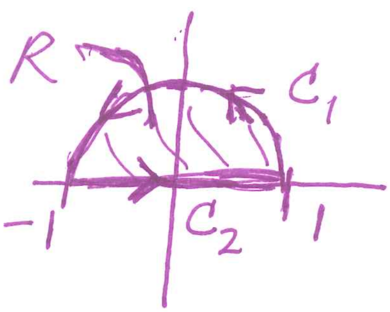
a) (3 points) Compute the 2-dimensional curl of $\mathbf{F}$.

$$Q_x - P_y = 3y^2e^x + 4 - (3y^2e^x) = 4.$$

ANSWER: 4

b) (7 points) Find the work done by $\mathbf{F}$ on a particle which moves along the top half of the circle of radius 1, centered at the origin, from $(1, 0)$ to $(-1, 0)$.

Use Green's Theorem.



$$\int_{C_1+C_2} \underline{F} \cdot d\underline{r} = \iint_R (Q_x - P_y)\, dA$$

$$= \iint_R 4\, dA = 4\begin{pmatrix}\text{area}\\ \text{of } R\end{pmatrix} = 4\left(\tfrac{1}{2}\pi(1)^2\right) = 2\pi.$$

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_{C_1+C_2} \underline{F} \cdot d\underline{r} - \int_{C_2} \underline{F} \cdot d\underline{r}$$

$$= 2\pi - \int_{C_2} \underline{F} \cdot d\underline{r} = 2\pi - \int_{C_2} (-x, 4x) \cdot (dx, dy)$$

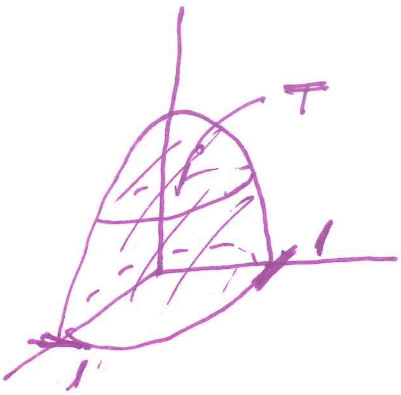
(dy = 0 along C_2)

$$= 2\pi - \int_{-1}^{1} -x\, dx = 2\pi + \left(\frac{x^2}{2}\Big|_{-1}^{1}\right) = 2\pi.$$

ANSWER: 2π joules

11) (8 points) Calculate the total outward flux of the vector field $\mathbf{F}(x, y, z) = (x^3, y^3, 1)$ across the boundary of the solid bounded by the paraboloid where $z = 1 - x^2 - y^2$ and the xy-plane.

Use the Divergence Theorem.



$$\iint_{\partial T} \underline{F} \cdot \underline{n} \, dS = \iiint_T (\vec{\nabla} \cdot \underline{F}) \, dV$$

$$= \iiint_T (3x^2 + 3y^2 + 0) \, dV$$

Use cylindrical coords.

$$= 3 \iiint_T (x^2 + y^2) \, dV$$

$$= 3 \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^2 \cdot r \, dz \, dr \, d\theta =$$

$$3 \int_0^{2\pi} \int_0^1 r^3 (1 - r^2) \, dr \, d\theta = 3 \cdot 2\pi \int_0^1 (r^3 - r^5) \, dr$$

$$= 6\pi \left(\frac{r^4}{4} - \frac{r^6}{6} \right) \Big|_0^1 = 6\pi \left(\frac{1}{4} - \frac{1}{6} \right) = \frac{\pi}{2}.$$

ANSWER: $\pi/2$

12) Let $\mathbf{F}(x,y,z) = (x^2 - z, x + y^2, z^2)$. Let C be the curve of intersection of the plane $z = 1 - x - y$ and the cylinder $x^2 + y^2 = 9$. Assume that C is oriented counterclockwise, as viewed from above. Let M be the region in the plane $z = 1 - x - y$ which is inside the curve C, so that C is the boundary of M.

a) (2 points) Calculate the curl of $\mathbf{F}(x,y,z)$.

$$\vec{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z & x + y^2 & z^2 \end{vmatrix} = (0, -(0-1), 1).$$

ANSWER: $\vec{\nabla} \times \underline{F} = (0, -1, 1).$

b) (2 points) Determine a unit normal to M which is compatible with the given orientation on its boundary C.

$\underline{n}$ is the upward-pointing unit normal to the plane where $x + y + z = 1$.

$$\underline{n} = \frac{(1,1,1)}{\sqrt{3}}.$$

ANSWER: $\underline{n} = \frac{(1,1,1)}{\sqrt{3}}.$

c) (4 points) Use Stokes' Theorem to calculate $\int_C \mathbf{F} \cdot dr$.

$$\int_C \underline{F} \cdot d\underline{r} = \iint_M (\vec{\nabla} \times \underline{F}) \cdot \underline{n} \, dS =$$

$$\iint_M (0,-1,1) \cdot \frac{(1,1,1)}{\sqrt{3}} \, dS = \iint_M 0 \, dS.$$

ANSWER: 0