1. If V is an inner product space and $\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{w}, \mathbf{x} \rangle$ for all $\mathbf{x} \in V$, prove that $\mathbf{v} = \mathbf{w}$.
2. Prove the Pythagorean theorem: If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors in an inner product and $\mathbf{c} = \mathbf{a} + \mathbf{b}$, then $||\mathbf{a}||^2 + ||\mathbf{b}||^2 = ||\mathbf{c}||^2$.
3. If $T : V \to V$ has an adjoint T^* and T^*T is the zero transformation, show that T is the zero transformation.
4. If V is finite-dimensional, show that $T : V \to V$ is invertible if and only if 0 is not an eigenvalue of T.
5. If $T : V \to V$ is invertible, show that λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.
6. If V is a complex vector space and $T : V \to V$ is diagonalizable and only has one eigenvalue λ, prove that T is the multiplication-by-λ map λI.
7. For any field F, prove that similarity is an equivalence relation on $M_{n\times n}(F)$.
8. For any $n \times n$ matrix A, prove that A and A^T have the same characteristic polynomial and hence the same eigenvalues.
9. Suppose $A \in M_{n\times n}(F)$ has two distinct eigenvalues λ and μ. If the λ-eigenspace has dimension $n-1$, prove that A is diagonalizable.
10. Prove that $T : V \to V$ is diagonalizable if and only if V is the direct sum of the eigenspaces of T.
11. If A is an $n \times n$ matrix, define the subspace $W \subseteq M_{n\times n}(F)$ as $W = \text{span}(I_n, A, A^2, A^3, \dots)$. Prove that $\dim_F W \le n$.
12. If $\beta = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a chain of generalized λ-eigenvectors (i.e., with $(T - \lambda I)\mathbf{v}_i = \mathbf{v}_{i+1}$ for each $0 \le i \le k-1$ and $(T - \lambda I)\mathbf{v}_k = \mathbf{0}$) show that β is linearly independent.
13. Prove that, up to similarity, there are exactly 5 different matrices in $M_{6\times 6}(\mathbb{C})$ with characteristic polynomial $p(t) = t^6 - t^4$.
14. Suppose $A \in M_{n\times n}(\mathbb{C})$ such that $A^2 - 2A = 0$. Prove that the Jordan canonical form of A must be diagonal.
15. Suppose $T : V \to V$ is diagonalizable. Prove that for any λ in the scalar field of V, it is true that $\text{rank}(T - \lambda I) = \text{rank}(T - \lambda I)^2$.
16. Suppose $T : V \to V$ has the property that $\text{rank}(T - \lambda I) = \text{rank}(T - \lambda I)^2$ for every λ in the scalar field of V. Prove that T is diagonalizable. [Hint: Consider the λ-blocks in the Jordan canonical form.]
17. If V is a complex inner product space and $T : V \to V$ is Hermitian, prove that $||T(\mathbf{v}) + i\mathbf{v}||^2 = ||T(\mathbf{v})||^2 + ||\mathbf{v}||^2$, and deduce that $T + iI$ is invertible. [Hint: Show $T + iI$ is one-to-one.]
18. If V is a finite-dimensional inner product space and $T : V \to V$ has an adjoint T^*, prove that all eigenvalues of T^*T are nonnegative real numbers, and deduce that $I + T^*T$ is invertible.