1. If S is a nonempty set, $x_0 \in S$ is fixed, F is a field, and V is the space of functions from S to F, show that the set $W = \{f \in V : f(x_0) = 0\}$ is a subspace of V.

2. If W_1 and W_2 are subspaces of V, show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

3. Show that the set of convergent real-valued sequences is a subspace of the vector space of all real-valued sequences.

4. Show that W is a subspace of V if and only if $W = \text{span}(W)$.

5. If $\text{char}(F) \neq 3$ and $S = \{\mathbf{v}, \mathbf{w}\}$ and $T = \{\mathbf{v} + \mathbf{w}, \mathbf{v} - 2\mathbf{w}\}$, show that (a) if S spans V then T spans V, (b) if S is linearly independent then T is linearly independent, and (c) if S is a basis then T is a basis.

6. Show that a vector space is infinite-dimensional if and only if it contains an infinite linearly-independent subset.

7. Show that any set of polynomials, no two of which have the same degree, is linearly independent.

8. Show that if $g \in P_n(F)$ has degree n and F has characteristic zero, then for any polynomial $f \in P_n(F)$ there exist scalars $a_0, \dots, a_n$ such that $f = a_0 g + a_1 g' + a_2 g'' + \cdots + a_n g^{(n)}$, where $g^{(n)}$ is the nth derivative of g.

9. Prove that if W_1 and W_2 are finite-dimensional then $\dim(W_1 \cap W_2) \leq \min(\dim W_1, \dim W_2)$ and $\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2)$.

10. Let $T : V \to W$ be linear. If V_1 is a subspace of V, we define the image $T(V_1)$ as $T(V_1) = \{T(\mathbf{v}_1) : \mathbf{v}_1 \in V_1\}$. Show that $T(V_1)$ is a subspace of W. If T is an isomorphism also show that T restricts to an isomorphism of V_1 with $T(V_1)$, and deduce that $\dim(T(V_1)) = \dim(V_1)$.

11. Let $T : V \to W$ be linear. If W_1 is a subspace of W, we define the inverse image $T^{-1}(W_1)$ as $T^{-1}(W_1) = \{\mathbf{v} \in V : T(\mathbf{v}) \in W_1\}$. Show that $T^{-1}(W_1)$ is a subspace of V. (Note that T^{-1} is not necessarily a function here.)

12. Let $S : V \to W$ and $T : U \to V$ be linear. Show that if S and T are both one-to-one then ST is one-to-one, and also if S and T are both onto then ST is also onto.

13. Let $T : V \to W$ be linear. If T is one-to-one, show that T maps linearly independent sets to linearly independent sets. If T is onto, show that T maps spanning sets to spanning sets.

14. Let $V = M_{n \times n}(F)$. If $B \in V$ is invertible, show that the map $\Phi_B : V \to V$ given by $\Phi_B(A) = B^{-1}AB$ is an isomorphism.