Math 4571 (Advanced Linear Algebra)

Lecture #30

Quadratic Forms (Part 1):

- Quadratic Forms
- Diagonalization of Quadratic Forms
- Quadratic Varieties in $\mathbb{R}^n$

This material represents §5.2.1 + §5.2.2 from the course notes.

In the previous lecture, we discussed bilinear forms on a vector space.

In particular, in the proof that symmetric forms are diagonalizable, the existence of a vector $\mathbf{x} \in V$ such that $\Phi(\mathbf{x}, \mathbf{x}) \neq 0$ played a central role. We now examine this (non-linear!) function $\Phi(\mathbf{x}, \mathbf{x})$ on V more closely.

Quadratic Forms, I

We begin by examining this associated form more closely:

Definition

If Φ is a symmetric bilinear form on V, the function $Q : V \to F$ with $Q(\mathbf{v}) = \Phi(\mathbf{v}, \mathbf{v})$ is called the <u>quadratic form</u> associated to Φ.

Quadratic Forms, I

We begin by examining this associated form more closely:

Definition

If Φ is a symmetric bilinear form on V, the function $Q : V \to F$ with $Q(\mathbf{v}) = \Phi(\mathbf{v}, \mathbf{v})$ is called the <u>quadratic form</u> associated to Φ.

<u>Example</u>:

- If Φ is the symmetric bilinear form with matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$

  over F^2, then the corresponding quadratic form has
  $Q(\begin{bmatrix} x \\ y \end{bmatrix}) = x^2 + 6xy + 4y^2$.

- The fact that this is a homogeneous quadratic function of the entries of the input vector is the reason for the name "quadratic form".

More Examples:

- More generally, if A is any $n \times n$ matrix, then the quadratic form associated to $\Phi_A(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T A \mathbf{w}$ is $Q_A(\mathbf{v}) = \mathbf{v}^T A \mathbf{v}$.

- So for example, if $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ 4 & 7 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then
  $Q_A(\mathbf{v}) = x^2 + 6xy + 8yz + 4y^2 + 14yz + 2z^2$.

- If Φ is an inner product $\langle \cdot, \cdot \rangle$ on a real vector space, then the associated quadratic form is $Q(\mathbf{v}) = ||\mathbf{v}||^2$, the square of the norm of $\mathbf{v}$.

Clearly, Q is uniquely determined by Φ. When $\mathrm{char}(F) \neq 2$, the reverse holds as well.

- Explicitly, since
  $Q(\mathbf{v} + \mathbf{w}) = \Phi(\mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) = Q(\mathbf{v}) + 2\Phi(\mathbf{v}, \mathbf{w}) + Q(\mathbf{w})$, we
  can write $\Phi(\mathbf{v}, \mathbf{w}) = \frac{1}{2}[Q(\mathbf{v} + \mathbf{w}) - Q(\mathbf{v}) - Q(\mathbf{w})]$, and so we
  may recover Φ from Q.

- Also, observe that for any scalar $\alpha \in F$, we have
  $Q(\alpha\mathbf{v}) = \Phi(\alpha\mathbf{v}, \alpha\mathbf{v}) = \alpha^2\Phi(\mathbf{v}, \mathbf{v}) = \alpha^2 Q(\mathbf{v})$.

- These last two relations provide us a way to define a quadratic
  form without explicit reference to the underlying symmetric
  bilinear form.

Definition

If V is a vector space, a __quadratic form__ is a function $Q : V \to F$ such that $Q(\alpha \mathbf{v}) = \alpha^2 Q(\mathbf{v})$ for all $\alpha \in F$, and the function $Q(\mathbf{v} + \mathbf{w}) - Q(\mathbf{v}) - Q(\mathbf{w})$ is a bilinear form in $\mathbf{v}$ and $\mathbf{w}$.

Definition

If V is a vector space, a <u>quadratic form</u> is a function $Q : V \to F$ such that $Q(\alpha\mathbf{v}) = \alpha^2 Q(\mathbf{v})$ for all $\alpha \in F$, and the function $Q(\mathbf{v} + \mathbf{w}) - Q(\mathbf{v}) - Q(\mathbf{w})$ is a bilinear form in $\mathbf{v}$ and $\mathbf{w}$.

Here are some basic properties (easy to see from the definition):

- Sensibly enough, the quadratic form associated to any bilinear form is a quadratic form (per the definition above).
- By setting $\alpha = 0$ we see $Q(\mathbf{0}) = 0$, and by setting $\alpha = -1$ we see $Q(-\mathbf{v}) = Q(\mathbf{v})$.
- Like with bilinear forms, the set of all quadratic forms on V forms a vector space.

<u>Example</u> (again): Show that the function
$Q[(x, y)] = x^2 + 6xy + 4y^2$ is a quadratic form on F^2.

Example (again): Show that the function
$Q[(x, y)] = x^2 + 6xy + 4y^2$ is a quadratic form on F^2.

- First, $Q[\alpha(x, y)] = (\alpha x)^2 + 6(\alpha x)(\alpha y) + 4(\alpha y)^2 = \alpha^2(x^2 + 6xy + 4y^2) = \alpha^2 Q(x, y)$.

- Also, $Q[(x_1, y_1) + (x_2, y_2)] - Q[(x_1, y_1)] - Q[(x_2, y_2)] = 2x_1 x_2 + 6x_1 y_2 + 6x_2 y_1 + 8y_1 y_2$. It is straightforward to verify that this is a bilinear form by checking the linearity explicitly.

- Alternatively (at least when $\mathrm{char}(F) \neq 2$) we can write down the associated bilinear form
  $\Phi((a, b), (c, d)) = \frac{1}{2}[Q[(a+c, b+d)] - Q[(a, c)] - Q[(b, d)]] = ac + 3ad + 3bc + 4bd$, and this is the bilinear form associated to the matrix $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$, as indeed we saw earlier.

<u>Example</u>: If $V = C[a, b]$, show that the function $Q(f) = \int_a^b f(x)^2 \, dx$ is a quadratic form on V.

<u>Example</u>: If $V = C[a, b]$, show that the function
$Q(f) = \int_a^b f(x)^2 \, dx$ is a quadratic form on V.

- First, we have
  $Q(\alpha f) = \int_a^b [\alpha f(x)]^2 \, dx = \alpha^2 \int_a^b f(x)^2 \, dx = \alpha^2 Q(f)$.
- Also, we have $Q(f + g) - Q(f) - Q(g) = \int_a^b [f(x) + g(x)]^2 \, dx - \int_a^b f(x)^2 \, dx - \int_a^b g(x)^2 \, dx = \int_a^b 2f(x)g(x) \, dx$, and this is indeed a bilinear form in f and g.

If $\operatorname{char}(F) \neq 2$, then the function $\dfrac{1}{2}[Q(\mathbf{v} + \mathbf{w}) - Q(\mathbf{v}) - Q(\mathbf{w})]$ is the bilinear pairing associated to Q.

- Indeed, the association maps give an isomorphism between the spaces of quadratic forms and bilinear pairings.
- In particular, any homogeneous quadratic function on F^n (i.e., any polynomial function all of whose terms have total degree 2) is a quadratic form on F^n.
- For variables $x_1, \ldots, x_n$, such a function has the general form $Q = \sum_{1 \leq i \leq j \leq n} a_{i,j} x_i x_j$.
- The associated matrix A for the corresponding bilinear form has entries $a_{i,j} = a_{j,i} = \begin{cases} a_{i,i} & \text{for } i = j \\ a_{i,j}/2 & \text{for } i \neq j \end{cases}$.

Quadratic Forms, VIII

Examples:

- The function $Q(x_1, x_2) = 7x_1^2 - 4x_1x_2 + 3x_2^2$ is a quadratic form on F^2. The matrix for the associated symmetric bilinear form is $\begin{bmatrix} 7 & -2 \\ -2 & 3 \end{bmatrix}$.

- The function $Q(x_1, x_2, x_3) = x_1^2 + 2x_1x_3 - 3x_2x_3 + 4x_3^2$ is a quadratic form on F^3. When $\mathrm{char}(F) \neq 2$, the matrix for the associated symmetric bilinear form is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -\frac{3}{2} \\ 1 & -\frac{3}{2} & 4 \end{bmatrix}$.

- The function $Q(x_1, \ldots, x_n) = x_1^2 + 2x_2^2 + 3x_3^2 + \cdots + nx_n^2$ is a quadratic form on F^n. Its associated matrix is the diagonal matrix with entries $1, 2, \ldots, n$.

We will now specialize to the specific case of quadratic forms on a finite-dimensional real vector space.

Our reason for this is that quadratic forms in this setting behave particularly nicely, and we will be able to describe several concrete applications to analysis and geometry (both of which naturally take place in $\mathbb{R}^n$.

In the situation where V is a finite-dimensional vector space over $F = \mathbb{R}$, by choosing a basis we may assume that $V = \mathbb{R}^n$ for concreteness.

We have discussed a procedure for diagonalization already, but we can give another one here:

- By the real spectral theorem, any real symmetric matrix is orthogonally diagonalizable, meaning that if S is any real symmetric matrix, then there exists an orthogonal matrix Q (with $Q^T = Q^{-1}$) such that $QSQ^{-1} = D$ is diagonal.

- Now, since $Q^T = Q^{-1}$, if we take $R = Q^T$ then this condition is the same as saying $R^T S R = D$ is diagonal.

- But this is precisely the condition we require in order to diagonalize a symmetric bilinear form!

- Hence: we may diagonalize a quadratic form over $\mathbb{R}$ by computing the (regular) diagonalization of the corresponding matrix: this is quite efficient as it only requires finding eigenvalues and eigenvectors.

One particular reason we will want to use the spectral approach to diagonalize a quadratic form over $\mathbb{R}^n$ is that the orthogonal change of basis has much nicer geometric properties.

- Specifically, the underlying diagonalization represents "completing the square" in the quadratic form via an orthogonal change of variables (i.e., one arising from an orthonormal basis).

- Geometrically, this corresponds to a rotation of the standard coordinate axes, possibly also with a reflection.

- Such transformations (as discussed on a past homework) are isometries, and so they preserve lengths and angles, which is not the case for other possible choices of diagonalization.

<u>Example</u>: Find an orthogonal change of basis that diagonalizes the quadratic form $Q(x, y, z) = 5x^2 + 4xy + 6y^2 + 4yz + 7z^2$ over $\mathbb{R}^3$.

- We simply diagonalize the matrix for the corresponding bilinear form, which is $A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$. The characteristic polynomial is $p(t) = \det(tI_3 - A) = t^3 - 18t^2 + 99t - 162 = (t - 3)(t - 6)(t - 9)$, so the eigenvalues are $\lambda = 3, 6, 9$.

- Computing a basis for each eigenspace yields eigenvectors $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ for $\lambda = 3, 6, 9$.

<u>Example</u>: Diagonalize $Q(x, y, z) = 5x^2 + 4xy + 6y^2 + 4yz + 7z^2$.

- Hence we may take $Q = \dfrac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$, so that

  $Q^T = Q^{-1}$ and $QAQ^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = D$.

- Therefore the desired change of basis is $x' = \dfrac{1}{3}(2x - 2y + z)$,
  $y' = \dfrac{1}{3}(-2x - y + 2z)$, $z' = \dfrac{1}{3}(x + 2y + 2z)$.

- With this change of basis it is not hard to verify that, indeed,
  $Q(x, y, z) = 3(x')^2 + 6(y')^2 + 9(z')^2$.

One application of the existence of such a diagonalization is to classify the conic sections in $\mathbb{R}^2$, and the quadric surfaces in $\mathbb{R}^3$. These curves (in $\mathbb{R}^2$) and surfaces (in $\mathbb{R}^3$)

- For conics in $\mathbb{R}^2$, the general equation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
- By diagonalizing, we may eliminate the xy term, and so the quadratic terms can be put into the form $Ax^2 + Cy^2$.
- We then have various cases depending on the signs of A and C. The end result (after doing some more algebra) is that every conic is either degenerate (e.g., a point, a line, two lines), or an ellipse, a parabola, or hyperbola.

Conics and Quadrics, II

Now we consider quadric surfaces in $\mathbb{R}^3$.

- Like with conics, we may eliminate cross-terms by diagonalizing, which yields a reduced equation $Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0$.

- We can then perform a similar analysis (based on how many of A, B, C are zero and the relative signs of the nonzero coefficients and the linear terms) to obtain all of the possible quadric surfaces in $\mathbb{R}^3$.

- In addition to the "degenerate" surfaces (e.g., a point, a plane, two planes), after rescaling the variables, one obtains 9 different quadric surfaces: the ellipsoid, the elliptic/parabolic/hyperbolic cylinders, the hyperboloid of one sheet, the elliptical cone, the hyperboloid of two sheets, the elliptic paraboloid, and the hyperbolic paraboloid.

It is worth using a computer graphing system, like Mathematica or Desmos, to plot all of these surfaces. Here are equations for the nine possible quadric surfaces in $\mathbb{R}^3$:

- Ellipsoid: $x^2 + y^2 + z^2 = 1$.
- Elliptic cylinder: $x^2 + y^2 = 1$.
- Parabolic cylinder: $y = x^2$.
- Hyperbolic cylinder: $x^2 - y^2 = 1$.
- Hyperboloid of two sheets: $x^2 + y^2 - z^2 = -1$.
- Elliptical cone: $x^2 + y^2 - z^2 = 0$.
- Hyperboloid of one sheet: $x^2 + y^2 - z^2 = 1$.
- Elliptic paraboloid $z = x^2 + y^2$.
- Hyperbolic paraboloid: $z = x^2 - y^2$.

All of the conics and quadric surfaces are examples of
<u>algebraic varieties</u>, which are the solution sets of polynomial
equations in several variables.

- If we have a general quadratic variety (i.e., a quadratic
  polynomial equation in n variables), we can make an
  appropriate translation and rescaling to convert it to the form
  $Q(x_1, \ldots, x_n) = 1$ or 0, where Q is a quadratic form.
- By diagonalizing the corresponding quadratic form using an
  orthonormal change of basis (which corresponds to a rotation
  of the coordinates axes and possibly also a reflection), we can
  then determine the shape of the variety's graph in $\mathbb{R}^n$.

Example: Diagonalize the quadratic form
$Q(x, y) = 2x^2 - 4xy - y^2$. Use the result to describe the shape of
the conic section $Q(x, y) = 1$ in $\mathbb{R}^2$.

Example: Diagonalize the quadratic form
$Q(x, y) = 2x^2 - 4xy - y^2$. Use the result to describe the shape of
the conic section $Q(x, y) = 1$ in $\mathbb{R}^2$.

- The matrix associated to the corresponding bilinear form is
  $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$.

- The characteristic polynomial is
  $p(t) = \det(tI_2 - A) = t^3 - t + 6$ with eigenvalues $\lambda = 3, -2$.

- We need to diagonalize A using an orthonormal basis of
  eigenvectors. Since the eigenvalues are distinct, we simply
  compute a basis for each eigenspace: doing so yields
  eigenvectors $(-2, 1)$ and $(1, 2)$ for $\lambda = 3, -2$ respectively.

<u>Example</u>: Describe the shape of $2x^2 - 4xy - y^2 = 1$ in $\mathbb{R}^2$.

- From the previous slide, we may diagonalize A via the orthogonal matrix $Q = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$, and the resulting diagonalization is $Q(x, y, z) = 3(x')^2 - 2(y')^2$.

- In particular, since the change of basis is orthonormal, in the new coordinate system the equation $Q(x, y, z) = 1$ reads simply as $3(x')^2 - 2(y')^2 = 1$.

- By rescaling again, with $x'' = \sqrt{3}x'$, $y'' = \sqrt{2}y'$, this is equivalent to $(x'')^2 - (y'')^2 = 1$, which is a hyperbola.

Example: Diagonalize the quadratic form
$Q(x, y, z) = 2x^2 + 4xy - 20xz + 11y^2 + 16yz + 5z^2$. Use the result
to describe the shape of the quadric surface $Q(x, y, z) = 1$ in $\mathbb{R}^3$.

<u>Example</u>: Diagonalize the quadratic form
$Q(x, y, z) = 2x^2 + 4xy - 20xz + 11y^2 + 16yz + 5z^2$. Use the result to describe the shape of the quadric surface $Q(x, y, z) = 1$ in $\mathbb{R}^3$.

- The matrix associated to the corresponding bilinear form is
$$A = \begin{bmatrix} 2 & 2 & -10 \\ 2 & 11 & 8 \\ -10 & 8 & 5 \end{bmatrix}.$$

- The characteristic polynomial is $p(t) = \det(tI_3 - A) = t^3 - 18t^2 - 81t + 1458 = (t + 9)(t - 9)(t - 18)$ with eigenvalues $\lambda = 9, 18, -9$.

- We need to diagonalize A using an orthonormal basis of eigenvectors. Since the eigenvalues are distinct, we simply compute a basis for each eigenspace: doing so yields eigenvectors $(-2, -2, 1)$, $(-1, 2, 2)$, $(2, -1, 2)$, for $\lambda = 9, 18, -9$ respectively.

Quadratic Varieties, IV

Example: Describe $2x^2 + 4xy - 20xz + 11y^2 + 16yz + 5z^2 = 1$.

- From the previous slide, we may diagonalize A via the orthogonal matrix $Q = \dfrac{1}{3} \begin{bmatrix} -2 & -1 & 2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix}$, and the resulting diagonalization is
  $Q(x, y, z) = 9(x')^2 + 18(y')^2 - 9(z')^2$.

- In particular, since the change of basis is orthonormal, in the new coordinate system the equation $Q(x, y, z) = 1$ reads simply as $9(x')^2 + 18(y')^2 - 9(z')^2 = 1$.

- By rescaling again, with $x'' = 3x'$, $y'' = 3\sqrt{2}y'$, $z'' = 3z'$, this is equivalent to $(x'')^2 + (y'')^2 - (z'')^2 = 1$, which is a $\boxed{\text{hyperboloid of one sheet}}$.

Summary

We introduced the notion of a quadratic form on a vector space, and discussed in detail the relationship between quadratic forms and symmetric bilinear forms.

We discussed diagonalization of quadratic forms, and in particular considerer the case of diagonalizing quadratic forms over $\mathbb{R}$.

We examined quadratic varieties in $\mathbb{R}^n$ and discussed how to use our results to classify them.

Next (and final!) lecture: Quadratic Forms (Part 2)