Math 4571 (Advanced Linear Algebra)

Lecture #29

Bilinear Forms:

- Bilinear Forms
- Associated Matrices and Congruence
- Diagonalization of Bilinear Forms

This material represents §5.1.1 + §5.1.2 from the course notes.

Overview

In this last segment of the course, we will pursue a brief introduction to bilinear forms and quadratic forms.

- Bilinear forms are simply linear transformations that are linear in two input variables, rather than just one.
- They are closely related to our other object of study: quadratic forms.
- Classically speaking, quadratic forms are homogeneous quadratic polynomials in multiple variables (e.g., $x^2 + 2xy + y^2$, or $xy + 2xz - y^2 - 3z^2$).
- The study of quadratic forms touches on nearly every branch of pure mathematics: linear algebra (as we will discuss, despite the fact that quadratic functions are avowedly not linear!), analysis and geometry (as we will also discuss), and even number theory and topology (as we will not discuss).

Bilinear Forms, I

We begin by defining bilinear forms.

Definition

A function $\Phi : V \times V \to F$ is a bilinear form on V if it is linear in each variable when the other variable is fixed. Explicitly, this means $\Phi(\mathbf{v}_1 + \alpha \mathbf{v}_2, y) = \Phi(\mathbf{v}_1, \mathbf{w}) + \alpha\Phi(\mathbf{v}_2, \mathbf{w})$ and $\Phi(\mathbf{v}, \mathbf{w}_1 + \alpha \mathbf{w}_2) = \Phi(\mathbf{v}, \mathbf{w}_1) + \alpha\Phi(\mathbf{v}, \mathbf{w}_2)$ for arbitrary $\mathbf{v}_i, \mathbf{w}_i \in V$ and $\alpha \in F$.

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<u>Examples</u>:

- If $V = F^2$, then $\Phi[(a, b), (c, d)] = ac + 2bc - ad + 3bd$ is a bilinear form on V.
- If $V = F^2$, then $\Phi[(a, b), (c, d)] = 2ac - bc + 4ad + 8bd$ is a bilinear form on V.

Bilinear Forms, II

More Examples:

- An inner product on a real vector space is a bilinear form, but an inner product on a complex vector space is not, since it is conjugate-linear in the second component rather than (actually) linear.
- Thus, for example, if $V = C[a, b]$ is the space of continuous functions on $[a, b]$, then $\Phi(f, g) = \int_a^b f(x)g(x)\,dx$ is a bilinear form on V (since it is an inner product).
- Likewise, the usual dot product on $\mathbb{R}^n$ is a bilinear form. More generally, the dot product on F^n is a bilinear form.
- The pairing $\Phi(A, B) = \mathrm{tr}(AB)$ on $M_{n\times n}(F)$ is a bilinear form.
- If $V = F[x]$ and $a, b \in F$, then $\Phi(p, q) = p(a)q(b)$ is a bilinear form on V.

Bilinear Forms, III

A large class of examples of bilinear forms arise as follows: if $V = F^n$, then for any matrix $A \in M_{n\times n}(F)$, the map $\Phi_A(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T A\mathbf{w}$ is a bilinear form on V.

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<u>Example</u>:

- The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ yields the bilinear form $\Phi_A\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = [x_1 \ y_1] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1x_2 + 2x_1y_2 + 3x_2y_1 + 4y_1y_2$.

Associated Matrices, I

In fact, if V is finite-dimensional, then by choosing a basis of V we can show that every bilinear form arises in the manner described on the previous slide.

Definition

If V is a finite-dimensional vector space, $\beta = \{\beta_1, \dots, \beta_n\}$ is a basis of V, and Φ is a bilinear form on V, the <u>associated matrix</u> of Φ with respect to β is the matrix $[\Phi]_\beta \in M_{n\times n}(F)$ whose (i,j)-entry is the value $\Phi(\beta_i, \beta_j)$.

This is the natural analogue of the matrix associated to a linear transformation, for bilinear forms.

Associated Matrices, II

Example: For the bilinear form $\Phi((a,b),(c,d)) = 2ac + 4ad - bc$ on F^2, find $[\Phi]_\beta$ for the standard basis $\beta = \{(1,0),(0,1)\}$.

Associated Matrices, II

<u>Example</u>: For the bilinear form $\Phi((a,b),(c,d)) = 2ac + 4ad - bc$ on F^2, find $[\Phi]_\beta$ for the standard basis $\beta = \{(1,0),(0,1)\}$.

- We simply calculate the four values $\Phi(\beta_i, \beta_j)$ for $i, j \in \{1,2\}$, where $\beta_1 = (1,0)$ and $\beta_2 = (0,1)$.
- This yields $\Phi(\beta_1,\beta_1) = 2$, $\Phi(\beta_1,\beta_2) = 4$, $\Phi(\beta_2,\beta_1) = -1$, and $\Phi(\beta_2,\beta_2) = 0$.
- Thus, the associated matrix is $[\Phi]_\beta = \boxed{\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}}$.

Associated Matrices, III

Example: For the bilinear form $\Phi((a, b), (c, d)) = 2ac + 4ad - bc$ on F^2, find $[\Phi]_\gamma$ for the basis $\gamma = \{(2, 1), (-1, 4)\}$.

Associated Matrices, III

Example: For the bilinear form $\Phi((a,b),(c,d)) = 2ac + 4ad - bc$ on F^2, find $[\Phi]_\gamma$ for the basis $\gamma = \{(2,1),(-1,4)\}$.

- We simply calculate the four values $\Phi(\gamma_i, \gamma_j)$ for $i, j \in \{1, 2\}$, where $\gamma_1 = (2,1)$ and $\gamma_2 = (-1,4)$.
- This yields $\Phi(\gamma_1,\gamma_1) = 14$, $\Phi(\gamma_1,\gamma_2) = 29$, $\Phi(\gamma_2,\gamma_1) = -16$, and $\Phi(\gamma_2,\gamma_2) = -10$.
- Thus, the associated matrix is $[\Phi]_\gamma = \boxed{\begin{bmatrix} 14 & 29 \\ -16 & -10 \end{bmatrix}}$.

Notice that, as one would expect, the associated matrix changes if we use a different basis.

Associated Matrices, IV

Example: For the bilinear form $\Phi(p,q) = \int_0^1 p(x)q(x)\,dx$ on $P_2(\mathbb{R})$, find $[\Phi]_\beta$ for the basis $\beta = \{1, x, x^2\}$.

Associated Matrices, IV

Example: For the bilinear form $\Phi(p, q) = \int_0^1 p(x)q(x)\,dx$ on $P_2(\mathbb{R})$, find $[\Phi]_\beta$ for the basis $\beta = \{1, x, x^2\}$.

- We simply calculate the nine values $\Phi(\beta_i, \beta_j)$ for $i, j \in \{1, 2, 3\}$, where $\beta_1 = 1$, $\beta_2 = x$, $\beta_3 = x^2$.
- For example, $\Phi(\beta_1, \beta_3) = \int_0^1 1 \cdot x^2\,dx = \dfrac{1}{3}$ and $\Phi(\beta_3, \beta_2) = \int_0^1 x^2 \cdot x\,dx = \dfrac{1}{4}$.
- The resulting associated matrix is

$$[\Phi]_\beta = \boxed{\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}}.$$

Properties of Associated Matrices, I

Let us now record some of the properties of associated matrices:

Proposition (Associated Matrices)

Suppose that V is a finite-dimensional vector space, $\beta = \{\beta_1, \dots, \beta_n\}$ is a basis of V, and Φ is a bilinear form on V. Then the following hold:

1. *If $\mathbf{v}$ and $\mathbf{w}$ are any vectors in V, then $\Phi(\mathbf{v}, \mathbf{w}) = [\mathbf{v}]_\beta^T [\Phi]_\beta [\mathbf{w}]_\beta$.*
2. *The map $\Phi \mapsto [\Phi]_\beta$ is an isomorphism of the space $\mathcal{B}(V)$ of bilinear forms on V with $M_{n\times n}(F)$; thus $\dim_F \mathcal{B}(V) = n^2$.*
3. *If Φ^T is the reverse form of Φ defined via $\Phi^T(\mathbf{v}, \mathbf{w}) = \Phi(\mathbf{w}, \mathbf{v})$, then $[\Phi^T]_\beta = [\Phi]_\beta^T$.*
4. *If γ is another basis of V and $Q = [I]_\beta^\gamma$ is the change-of-basis matrix from β to γ, then $[\Phi]_\gamma = Q^T [\Phi]_\beta Q$.*

Properties of Associated Matrices, II

Proofs:

1. If $\mathbf{v}$ and $\mathbf{w}$ are any vectors in V, then $\Phi(\mathbf{v}, \mathbf{w}) = [\mathbf{v}]_\beta^T [\Phi]_\beta [\mathbf{w}]_\beta$.
   - Proof: If $\mathbf{v} = \beta_i$ and $\mathbf{w} = \beta_j$ then the result is immediate from the definition of matrix multiplication and the matrix $[\Phi]_\beta$.
   - The result for arbitrary $\mathbf{v}$ and $\mathbf{w}$ then follows by linearity.
2. The map $\Phi \mapsto [\Phi]_\beta$ is an isomorphism of the space $\mathcal{B}(V)$ of bilinear forms on V with $M_{n \times n}(F)$; thus $\dim_F \mathcal{B}(V) = n^2$.
   - Proof: We can define an inverse map as follows: given a matrix $A \in M_{n \times n}(F)$, define a bilinear form Φ_A via $\Phi_A(\mathbf{v}, \mathbf{w}) = [\mathbf{v}]_\beta^T A [\mathbf{w}]_\beta$.
   - It is easy to verify that this map is a well-defined linear transformation and that it is inverse to the map given above.
   - The dimension calculation is immediate.

Properties of Associated Matrices, III

Proofs:

3. If Φ^T is the reverse form of Φ defined via $\Phi^T(\mathbf{v}, \mathbf{w}) = \Phi(\mathbf{w}, \mathbf{v})$, then $[\Phi^T]_\beta = [\Phi]_\beta^T$.
   - Proof: By definition we have $\Phi^T(\mathbf{v}, \mathbf{w}) = [\mathbf{w}]_\beta^T [\Phi]_\beta [\mathbf{v}]_\beta$.
   - Since the matrix product on the right is a scalar, it is equal to its transpose, which is $[\mathbf{v}]_\beta^T [\Phi]_\beta^T [\mathbf{w}]_\beta$.
   - This means $[\Phi^T]_\beta$ and $[\Phi]_\beta^T$ agree, as bilinear forms, on all pairs of vectors $[\mathbf{v}]_\beta$ and $[\mathbf{w}]_\beta$ in F^n, so they are equal.
4. If γ is another basis of V and $Q = [I]_\beta^\gamma$ is the change-of-basis matrix from β to γ, then $[\Phi]_\gamma = Q^T [\Phi]_\beta Q$.
   - Proof: By definition, $[\mathbf{v}]_\gamma = Q[\mathbf{v}]_\beta$.
   - Hence $[\mathbf{v}]_\beta^T Q^T [\Phi]_\beta Q [\mathbf{w}]_\beta = [\mathbf{v}]_\gamma^T [\Phi]_\beta [\mathbf{w}]_\gamma$.
   - This means that $Q^T [\Phi]_\beta Q$ and $[\Phi]_\gamma$ agree, as bilinear forms, on all pairs of vectors $[\mathbf{v}]_\beta$ and $[\mathbf{w}]_\beta$ in F^n, so they are equal.

Congruence and Diagonalization, I

The last result of the proposition above tells us how bilinear forms behave under change of basis: rather than the more familiar conjugation relation $B = QAQ^{-1}$, we instead have a slightly different relation $B = Q^T AQ$.

Definition

If $A, B \in M_{n\times n}(F)$, we say that A is congruent to B if there exists an invertible $Q \in M_{n\times n}(F)$ such that $B = Q^T AQ$.

The matrices B and C are congruent if and only if they represent the same bilinear form in different bases.

Specifically, the translation is $B = [\Phi]_\beta$ and $C = [\Phi]_\gamma$, with $Q = [I]_\beta^\gamma$ being the corresponding change-of-basis matrix.

Congruence and Diagonalization, II

Just like with matrices, we can also ask: for which basis β will the associated matrix to the bilinear form Φ be as simple as possible?

- The matrix version of this question is: given a matrix B, what is the simplest matrix C to which B is congruent?
- With matrices, the simplest possible answer is a diagonal matrix: that is of course still the simplest possible answer here.

So now we investigate which bilinear forms can be diagonalized.

Congruence and Diagonalization, III

Definition

If V is finite-dimensional, a bilinear form Φ on V is diagonalizable if there exists a basis β of V such that $[\Phi]_\beta$ is a diagonal matrix.

Congruence and Diagonalization, III

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If V is finite-dimensional, a bilinear form Φ on V is diagonalizable if there exists a basis β of V such that $[\Phi]_\beta$ is a diagonal matrix.

Extremely Important Warning:

Although we use the same word, diagonalizability for bilinear forms is *not* the same as diagonalizability for linear transformations!

Make sure to keep straight the difference between the corresponding matrix versions: two matrices are similar when we can write $B = Q^{-1}AQ$, whereas they are congruent when we can write $B = Q^T AQ$.

Congruence and Diagonalization, IV

Examples:

- The bilinear form $\Phi[(a,b),(c,d)] = ac + 4bd$ is diagonalizable using the standard basis $\beta = \{(1,0),(0,1)\}$ since $[\Phi]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is a diagonal matrix.
- The bilinear form $\Phi[(a,b),(c,d)] = 2ac - 2ad - 2bc - bd$ is diagonalizable. If we use the standard basis β, the associated matrix is $[\Phi]_\beta = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$, which is not diagonal. However, using instead the basis $\gamma = \{(1,2),(2,-1)\}$, we obtain $[\Phi]_\beta = \begin{bmatrix} -10 & 0 \\ 0 & 15 \end{bmatrix}$, which is diagonal.

Diagonalizability and Symmetry, I

It turns out that when $\mathrm{char}(F) \neq 2$, there is an easy criterion for diagonalizability.

Definition

A bilinear form Φ on V is symmetric if $\Phi(\mathbf{v}, \mathbf{w}) = \Phi(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$. Equivalently, Φ is symmetric when $\Phi^T = \Phi$.

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Theorem (Diagonalization of Bilinear Forms)

Let V be a finite-dimensional vector space over a field F of characteristic not equal to 2. Then a bilinear form on V is diagonalizable if and only if it is symmetric.

Diagonalizability and Symmetry, II

Proof:

- For the forward direction, by taking associated matrices, we see immediately that if V is finite-dimensional with basis β, then Φ is a symmetric bilinear form if and only if $[\Phi]_\beta$ is equal to its transpose, which is to say, when it is a symmetric matrix.
- Now observe that if Φ is diagonalizable, then $[\Phi]_\beta$ is a diagonal matrix hence symmetric, and thus Φ must be symmetric. This yields the forward direction.

Diagonalizability and Symmetry, III

Proof (continued):

- For the reverse direction, we use induction on $n = \dim_F V$. The base case $n = 1$ is trivial.
- Now suppose the result holds for all spaces of dimension less than n, and let Φ be symmetric on V.
- If Φ is the zero form, then clearly Φ is diagonalizable. Otherwise, suppose Φ is not identically zero: we claim there exists a vector $\mathbf{x}$ with $\Phi(\mathbf{x}, \mathbf{x}) \neq 0$.
- By hypothesis, Φ is not identically zero so suppose that $\Phi(\mathbf{v}, \mathbf{w}) \neq 0$. If $\Phi(\mathbf{v}, \mathbf{v}) \neq 0$ or $\Phi(\mathbf{w}, \mathbf{w}) \neq 0$ we may take $\mathbf{x} = \mathbf{v}$ or $\mathbf{x} = \mathbf{w}$. Otherwise, we have $\Phi(\mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}) = \Phi(\mathbf{v}, \mathbf{v}) + 2\Phi(\mathbf{v}, \mathbf{w}) + \Phi(\mathbf{w}, \mathbf{w}) = 2\Phi(\mathbf{v}, \mathbf{w}) \neq 0$ by the assumption that $\Phi(\mathbf{v}, \mathbf{w}) \neq 0$ and $2 \neq 0$ in F (here is where we require the characteristic not to equal 2), and so we may take $\mathbf{x} = \mathbf{v} + \mathbf{w}$.

Diagonalizability and Symmetry, IV

Proof (continued more):

- Now that we have $\mathbf{x}$ with $\Phi(\mathbf{x}, \mathbf{x}) \neq 0$, consider the linear functional $T : V \to F$ given by $T(\mathbf{v}) = \Phi(\mathbf{x}, \mathbf{v})$.
- Since $T(\mathbf{x}) = \Phi(\mathbf{x}, \mathbf{x}) \neq 0$, we see that $\mathrm{im}(T) = F$, so $\dim_F \ker(T) = n - 1$ by the nullity-rank theorem.
- Then the restriction of Φ to $\ker(T)$ is clearly a symmetric bilinear form on $\ker(T)$, so by induction, there exists a basis $\{\beta_1, \dots, \beta_{n-1}\}$ of $\ker(T)$ such that the restriction of Φ is diagonalized by this basis, so that $\Phi(\beta_i, \beta_j) = 0$ for $i \neq j$.
- Now set $\beta_n = \mathbf{x}$ and observe that since $\mathbf{x} \notin \ker(T)$, the set $\beta = \{\beta_1, \dots, \beta_{n-1}, \beta_n\}$ is a basis of V.
- Since $\Phi(\mathbf{x}, \beta_i) = \Phi(\beta_i, \mathbf{x}) = 0$ for all $i < n$ by definition of T, we conclude that β diagonalizes Φ, as required.

Diagonalizability and Symmetry, V

We will note that the assumption that $\mathrm{char}(F) \neq 2$ in the theorem above cannot be removed.

- Explicitly, if $F = \mathbb{F}_2$ is the field with 2 elements, then if Φ is the bilinear form on F^2 with associated matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then Φ is symmetric but cannot be diagonalized.
- Explicitly, suppose $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: then $Q^T A Q =$ $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & ad+bc \\ ad+bc & 0 \end{bmatrix}$, so the only possible diagonalization of Φ would be the zero matrix, but that is impossible because Φ is not the zero form.
- In this example we can see that $\Phi(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in F^2$, which causes the inductive argument to fail.

Diagonalizability and Symmetry, V

As an immediate corollary, we see that every symmetric matrix is congruent to a diagonal matrix in characteristic $\neq 2$:

Corollary

If $\mathrm{char}(F) \neq 2$, then every symmetric matrix over F is congruent to a diagonal matrix.

Proof: The result follows immediately by diagonalizing the corresponding bilinear form.

Computing Diagonalizations, I

In principle, the proof we gave is entirely constructive, since it provides a recursive way to generate a diagonalizing basis β. However, as a practical matter, we would like to have a more computationally useful algorithm.

In fact, we can give an explicit procedure for writing a symmetric matrix S in the form $D = Q^T S Q$ that is similar to the algorithm for computing the inverse of a matrix.

Computing Diagonalizations, II

We will use some facts about elementary row and column operations on matrices:

- Recall that if E is an elementary row matrix (obtained by performing an elementary row operation on the identity matrix), then EA is the matrix obtained by performing that elementary row operation on A.
- Likewise, if C is an elementary column matrix, then AC is the matrix obtained by performing that elementary column operation on A.
- Hence if E is an elementary row matrix, then EAE^T is the matrix obtained by performing the elementary row operation on A (given by E) and then the corresponding elementary column operation (given by E^T).

Computing Diagonalizations, III

Now suppose that $D = Q^T SQ$ where Q is invertible.

- Since Q is invertible, it is a product $E_1 \cdots E_d$ of elementary row matrices by our results on row-reduction.
- Then $Q^T SQ = E_d^T \cdots E_1^T SE_1 \cdots E_d$ is obtained from S by performing a sequence of these paired row-column operations.
- Our theorem on diagonalization above ensures that there is a sequence of these operations that will yield a diagonal matrix.

Computing Diagonalizations, IV

We may find the proper sequence of operations by performing these "paired" operations using a method similar to row-reduction:

- Using the (1,1)-entry, we apply row operations to clear out all the entries in the first column below it. (If this entry is zero, we add an appropriate multiple of another row to the top row to make it nonzero.)
- This will also clear out the column entries to the right of the (1,1)-entry, yielding a matrix whose first row and column are now diagonalized.
- Now restrict attention to the smaller $(n-1) \times (n-1)$ matrix excluding the first row and column, and repeat the procedure recursively until the matrix is diagonalized.
- Then we may obtain the matrix $Q^T = E_d^T \cdots E_1^T I$ by applying all of the elementary row operations (in the same order) starting with the identity matrix.

Computing Diagonalizations, V

We may keep track of these operations using a “double matrix” as in the algorithm for computing the inverse of a matrix: on the left we start with the symmetric matrix S, and on the right we start with the identity matrix I.

- At each step, we select a row operation and perform it, and its corresponding column operation, on the left matrix. We also perform the row operation (but *only* the row operation!) on the right matrix.
- When we are finished, we will have transformed the double-matrix $[S|I]$ into the double-matrix $[D|Q^T]$, and we will have $Q^T SQ = D$.

Computing Diagonalizations, VI

Example: For $S = \begin{bmatrix} 1 & 3 \\ 3 & -4 \end{bmatrix}$, find an invertible matrix Q and diagonal matrix D such that $Q^T S Q = D$.

Computing Diagonalizations, VI

<u>Example</u>: For $S = \begin{bmatrix} 1 & 3 \\ 3 & -4 \end{bmatrix}$, find an invertible matrix Q and diagonal matrix D such that $Q^T S Q = D$.

- We set up the double matrix and perform row/column operations as listed (to emphasize again, we perform the row and then the corresponding column operation on the left side, but *only the row operation* on the right side):

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right] \xrightarrow[C_2 - 3C_1]{R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -13 & -3 & 1 \end{array}\right]$$

- The matrix on the left is now diagonal.
- Thus, we may take $D = \begin{bmatrix} 1 & 0 \\ 0 & -13 \end{bmatrix}$ with $Q^T = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and thus $Q = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.
- Indeed, one may double-check that $Q^T S Q = D$, as claimed.

Computing Diagonalizations, VII

Example: For $S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$, find an invertible matrix Q and diagonal matrix D such that $Q^T S Q = D$.

- We set up the double matrix and do row/column operations:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_2 - 2C_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 3 & -6 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_3 - 3C_1]{R_3 - 3R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -7 & -3 & 0 & 1 \end{array}\right] \xrightarrow[C_3 - 2C_2]{R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 1 & 0 \\ 0 & 0 & 5 & 1 & -2 & 1 \end{array}\right]$$

Computing Diagonalizations, VIII: Is Enough

Example: (continued)

- The last matrix was

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 1 & 0 \\ 0 & 0 & 5 & 1 & -2 & 1 \end{array}\right]$$

- The matrix on the left is now diagonal.
- Thus, we may take $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ with

$Q^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and thus $Q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

Indeed, one may double-check that $Q^T S Q = D$, as claimed.

Summary

We introduced the notion of a bilinear form on a vector space, and established some of their basic properties.

We discussed the matrix associated to a bilinear form and how change of basis leads to the congruence relation on matrices.

We discussed diagonalization of bilinear forms, proved that a bilinear form is diagonalizable if and only if it is symmetric, and gave an algorithm for computing diagonalizations.

Next lecture: Quadratic Forms (Part 1)