Math 4571 (Advanced Linear Algebra)

Lecture #28

Applications of Diagonalization and the Jordan Canonical Form (Part 2):

- Systems of Linear Differential Equations
- The Eigenvalue Method
- Matrix Exponentials

This material represents §4.4.2 + §4.4.3 from the course notes.

Systems of Differential Equations, I

In this lecture, we discuss how to use diagonalization and the Jordan canonical form to solve systems of ordinary linear differential equations with constant coefficients. We begin by outlining the basic setup and terminology:

- A differential equation is merely an equation involving a derivative (or several derivatives) of a function or functions, such as $y' + y = 0$, $y'' + z'' = x^2 y'$, or $\ln(y') - x\sin(2y) = \pi$.
- Most differential equations are hard to solve[*citation needed*].
- We treat systems of ordinary linear differential equations: there is one independent variable x with various functions $y_1(x), \dots, y_n(x)$ to be determined, and each equation is linear in the y_i with functions of x as the coefficients.
- For example, $y'' + xy' + e^x y = 0$ is a linear differential equation, but $yy' = x$ is not since it contains the product yy'.

Systems of Differential Equations, II

We first observe that given any system of differential equations, we can convert it to a system of first-order differential equations (involving only first derivatives) by defining new variables.

- For example, we can convert the single equation $y''' + x^2 y' = 0$ into a system of first-order equations by defining new variables $z = y'$ and $w = y'' = z'$.
- Then observe $w' = y''' = -x^2 y'$.
- Hence, the single equation $y''' + x^2 y' = 0$ is equivalent to the system $y' = z$, $z' = w$, $w' = -x^2 z$.
- Notice that each equation now only involves first derivatives.

Systems of Differential Equations, III

By rearranging the equations and defining new variables appropriately we can put any system of linear differential equations into the form

$$\begin{aligned} y_1' &= a_{1,1}(x) \cdot y_1 + a_{1,2}(x) \cdot y_2 + \cdots + a_{1,n}(x) \cdot y_n + q_1(x) \\ \vdots \quad & \quad \vdots \\ y_n' &= a_{n,1}(x) \cdot y_1 + a_{n,2}(x) \cdot y_2 + \cdots + a_{n,n}(x) \cdot y_n + q_n(x) \end{aligned}$$

for some functions $a_{i,j}(x)$ and $q_i(x)$ for $1 \leq i, j \leq n$.

Systems of Differential Equations, IV

We can write this system more compactly using matrices: if

$$A = \begin{bmatrix} a_{1,1}(x) & \cdots & a_{1,n}(x) \\ \vdots & \ddots & \vdots \\ a_{n,1}(x) & \cdots & a_{n,n}(x) \end{bmatrix}, \ \mathbf{q} = \begin{bmatrix} q_1(x) \\ \vdots \\ q_n(x) \end{bmatrix}, \text{ and}$$

$$\mathbf{y} = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix} \text{ so that } \mathbf{y}' = \begin{bmatrix} y_1'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \text{ we can write the system}$$

(vastly!) more compactly as

$$\mathbf{y}' = A\mathbf{y} + \mathbf{q}.$$

Note the similarities to our use of matrices to solve systems of linear equations.

Systems of Differential Equations, V

We say that the system is homogeneous if $\mathbf{q} = \mathbf{0}$, and it is nonhomogeneous otherwise.

- Just like with systems of linear equations, the solution space to a homogeneous system forms a vector space (indeed, it is the kernel of an appopriate linear transformation).
- Solving a homogeneous system, therefore, amounts to finding a basis for this solution space.
- Finding the general solution to the system then requires only finding a single solution to the nonhomogeneous equation, and adding the general homogeneous solution to it.

Systems of Differential Equations, VI

As mentioned previously, it is quite hard to solve general systems. We will now restrict to the special case of homogeneous systems whose coefficients are constants (i.e., scalars). In this case we have the following preliminary result:

Theorem (Homogeneous Systems)

If the $n \times n$ coefficient matrix A is constant and I is any interval, then the set of solutions $\mathbf{y}$ to the homogeneous system $\mathbf{y}' = A\mathbf{y}$ on I is an n-dimensional vector space.

The proof of this theorem is a standard result from the analysis of differential equations. (The underlying field can be $\mathbb{R}$ or $\mathbb{C}$.)

Eigenvalue Method, I

All of this is very nice, but does not really help us solve actual systems. Our starting point is the following observation:

Observation

If $\mathbf{v} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ *is an eigenvector of* A *with eigenvalue* λ*, then* $\mathbf{y} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} e^{\lambda x}$ *is a solution to* $\mathbf{y}' = A\mathbf{y}$.

To see this just differentiate $\mathbf{y} = e^{\lambda x}\mathbf{v}$ with respect to x: this yields $\mathbf{y}' = \lambda e^{\lambda x}\mathbf{v} = \lambda \mathbf{y} = A\mathbf{y}$.

Eigenvalue Method, II

Example: For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, find a solution to $\mathbf{y}' = A\mathbf{y}$.

Eigenvalue Method, II

Example: For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, find a solution to $\mathbf{y}' = A\mathbf{y}$.

- Observe $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.
- We can write the system $\mathbf{y}' = A\mathbf{y}$ explicitly as

$$\begin{aligned} y_1' &= 2y_1 + 3y_2 \\ y_2' &= y_1 + 4y_2. \end{aligned}$$

- So we claim $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^t$ is a solution to the system.
- Indeed, if $y_1 = 3e^t$ and $y_2 = -e^t$, then in fact $y_1' = 3e^t = 2y_1 + 3y_2$ and $y_2' = -e^t = y_1 + 4y_2$, as claimed.

Eigenvalue Method, III

For each eigenvector, we obtain a solution to the system.

- In the event that A has n linearly independent eigenvectors (which is to say, if A is diagonalizable), we will therefore obtain n solutions to the differential equation.
- If these solutions are linearly independent, then since we know the solution space is n-dimensional, we would be able to conclude that our solutions are a basis for the solution space. This turns out to be true.

Theorem (Eigenvalue Method)

If A has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the general solution to the matrix differential system $\mathbf{y}' = A\mathbf{y}$ is given by $\mathbf{y} = C_1 e^{\lambda_1 x}\mathbf{v}_1 + C_2 e^{\lambda_2 x}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_2$, where $C_1, \cdots, C_n$ are arbitrary constants.

Eigenvalue Method, IV

Proof:

- Since the solution space is n-dimensional, we need only show that these vectors are linearly independent.
- To do this, we calculate the Wronskian W, the determinant of the matrix whose ith column is the vector $(y_i, y_i', y_i'', \dots, y_i^{(n-1)})$. If the functions were linearly dependent, then the rows would be linearly dependent and so the determinant would be zero.
- After factoring out the exponentials from each column, we obtain $W = e^{(\lambda_1 + \cdots + \lambda_n)x} \det(M)$, where M is the matrix whose columns are the eigenvectors $\mathbf{v}_i$.
- Since the eigenvectors are linearly independent, $\det(M) \neq 0$. This means $W \neq 0$ so the solution vectors are linearly independent, hence are a basis.

Eigenvalue Method, V

Thus, if the coefficient matrix is diagonalizable, we can produce the general solution to the system using this method.

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & y_1 - 3y_2 \\ y_2' & = & y_1 + 5y_2 \end{array} \right\}$.

Eigenvalue Method, V

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- We compute the eigenvectors of the coefficient matrix.
- With $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$ we get eigenvectors $\begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to the two eigenvalues $\lambda = 2, 4$ respectively.
- Thus, the general solution to the system is $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4x}$.
- Equivalently, $y_1 = -3C_1e^{2x} - C_2e^{4x}$, $y_2 = C_1e^{2x} + C_2e^{4x}$.

Eigenvalue Method, VI

If the real coefficient matrix has nonreal eigenvalues, by taking appropriate linear combinations and using Euler's identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ we can produce real-valued solutions.

- More explicitly, suppose that A has a nonreal eigenvalue $\lambda = a + bi$ with eigenvector $\mathbf{v} = \mathbf{w}_1 + i\mathbf{w}_2$.
- Then $\overline{\lambda} = a - bi$ will have eigenvector $\overline{\mathbf{v}} = \mathbf{w}_1 - i\mathbf{w}_2$.
- We see $\frac{1}{2}(e^{\lambda x}\mathbf{v} + e^{\overline{\lambda}x}\overline{v}) = e^{ax}(\mathbf{w}_1\cos(bx) - \mathbf{w}_2\sin(bx))$ and $\frac{1}{2i}(e^{\lambda x}\mathbf{v} - e^{\overline{\lambda}x}\overline{v}) = e^{ax}(\mathbf{w}_1\sin(bx) + \mathbf{w}_2\cos(bx))$ are then both real-valued, and have the same span.

Eigenvalue Method, VI

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & 2y_1 - y_2 \\ y_2' & = & y_1 + 2y_2 \end{array} \right\}$.

Eigenvalue Method, VI

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & 2y_1 - y_2 \\ y_2' & = & y_1 + 2y_2 \end{array} \right\}$.

- We compute the eigenvectors of the coefficient matrix.
- With $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ we get eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$, $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ corresponding to the two eigenvalues $\lambda = 2 \pm i$ respectively.
- The general solution is
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(2+i)x} + C_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(2-i)x}.$$
- Replacing the complex-valued solutions with real-valued ones yields an equivalent form
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = D_1 \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} e^{2x} + D_2 \begin{bmatrix} -\cos(x) \\ \sin(x) \end{bmatrix} e^{2x}.$$

Eigenvalue Method, VII

If the coefficient matrix is not diagonalizable, we cannot construct a basis for the solution space using the method we have described. Perhaps unsurprisingly, there is a way to fill in the missing basis elements using generalized eigenvectors, which we merely outline:

- Suppose we have a chain of generalized λ-eigenvectors $\{\mathbf{v}_k, \dots, \mathbf{v}_1, \mathbf{v}_0\}$, where $\mathbf{v}_{i-1} = (T - \lambda I)\mathbf{v_i}$.
- We leave it as an algebra exercise to show that the elements $e^{\lambda x}[\mathbf{v}_0]$, $e^{\lambda x}[x\mathbf{v}_0 + \mathbf{v}_1]$, $e^{\lambda x}[\frac{x^2}{2}\mathbf{v}_0 + x\mathbf{v}_1 + \mathbf{v}_2]$, ... , $e^{\lambda x}[\frac{x^k}{k!}\mathbf{v}_0 + \frac{x^{k-1}}{(k-1)!}\mathbf{v}_1 + \cdots + \mathbf{v}_k]$, are linearly independent solutions to the system.
- By our results on constructing the Jordan canonical form, we can always construct a chain basis for F^n. Then the vectors described above will give a corresponding basis for the solution space.

Eigenvalue Method, VIII

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & 5y_1 - 9y_2 \\ y_2' & = & 4y_1 - 7y_2 \end{array} \right\}$.

Eigenvalue Method, VIII

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & 5y_1 - 9y_2 \\ y_2' & = & 4y_1 - 7y_2 \end{array} \right\}$.

- The coefficient matrix $A = \begin{bmatrix} 5 & -9 \\ 4 & -7 \end{bmatrix}$ has eigenvalues $\lambda = -1, -1$ and is not diagonalizable.
- One may check that $\mathbf{v}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ give a chain basis for A.
- Thus, applying the formula yields the general solution $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-x} + C_2 \left[\begin{bmatrix} 3 \\ 2 \end{bmatrix} xe^{-x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-x} \right]$.

Matrix Exponentials, I

We now describe another (initially quite different, but later quite similar) method for using diagonalization and the Jordan canonical form to solve a homogeneous system of linear differential equations with constant coefficients.

- As motivation, if we consider the differential equation $y' = ky$ with the initial condition $y(0) = C$, it is not hard to verify that the general solution is $y(x) = e^{kx}C$.
- We would like to find some way to extend this result to an $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{c}$.
- The natural way would be to try to define the "exponential of a matrix" e^A in such a way that e^{At} has the property that $\frac{d}{dt}[e^{At}] = Ae^{At}$: then $\mathbf{y}(t) = e^{At}\mathbf{c}$ will have $\mathbf{y}'(t) = Ae^{At}\mathbf{c} = A\mathbf{y}$.

Matrix Exponentials, II

In fact, we can simply write down a definition for the exponential of a matrix as a power series:

Definition (Matrix Exponential)

If $A \in M_{n\times n}(\mathbb{C})$, we define the exponential of A, denoted e^A, to be the infinite sum $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$.

- The definition is motivated by the Taylor series for the exponential function; namely, $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
- In order for this definition to make sense, we need to know that the infinite sum actually converges. (Spoiler: it does.)

Matrix Exponentials, III

Theorem (Exponential Solutions)

For any matrix A, the infinite series $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges absolutely, in the sense that the series in each of the entries of the matrix converges absolutely. Furthermore, the unique solution to the initial value problem $\mathbf{y}' = A\mathbf{y}$ with $\mathbf{y}(a) = \mathbf{y}_0$ is given by $\mathbf{y}(t) = e^{A(t-a)}\mathbf{y}_0$.

The idea of the proof is to bound the sizes of the entries in the infinite sum and show that each sum converges. The second part then follows by differentiating term-by-term and comparing the new series to the original. (The full proof is in the notes.)

Matrix Exponentials, IV

The theorem on the previous slide says that we can use matrix exponentials to solve initial value problems. Of course, this requires actually being able to evaluate the matrix exponential, which we now describe.

- When the matrix is diagonalizable, we can do this easily: explicitly, if $A = Q^{-1}DQ$, then $e^A = Q^{-1}e^D Q$ (just conjugate all the terms in the power series).
- Furthermore, again from the power series definition, if $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, then $e^D = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$.
- Thus, by using the diagonalization, we can compute the exponential of the original matrix A, and thereby use it to solve the differential equation $\mathbf{y}' = A\mathbf{y}$.

Matrix Exponentials, V

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & 2y_1 - y_2 \\ y_2' & = & -2y_1 + 3y_2 \end{array} \right\}$.

Matrix Exponentials, V

Example: Find all functions y_1, y_2 with $\left\{ \begin{array}{rcl} y_1' & = & 2y_1 - y_2 \\ y_2' & = & -2y_1 + 3y_2 \end{array} \right\}$.

- The coefficient matrix $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ has distinct eigenvalues $\lambda = 1, 4$ so it is diagonalizable: with $Q = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, we see $Q^{-1}AQ = D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.
- Thus, $e^{At} = Qe^{Dt}Q^{-1} = Q \begin{bmatrix} e^t & 0 \\ 0 & e^{4t} \end{bmatrix} Q^{-1} = \frac{1}{3} \begin{bmatrix} 2e^t + e^{4t} & e^t - e^{4t} \\ 2e^t - 2e^{4t} & e^t + 2e^{4t} \end{bmatrix}$.
- Then $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2e^t + e^{4t} & e^t - e^{4t} \\ 2e^t - 2e^{4t} & e^t + 2e^{4t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ is the general solution, for arbitrary constants C_1 and C_2.

Matrix Exponentials, VI

Much like the situation with the eigenvalue method, things are more difficult if the matrix is not diagonalizable.

- We must replace the diagonalization with the Jordan canonical form, and it is a bit less obvious how to exponentiate a non-diagonal matrix.
- Recall that if $A = Q^{-1}JQ$, then $e^A = Q^{-1}e^J Q$. Then, since exponentiation works block-by-block in a block-diagonal matrix, it is enough to describe how to compute the exponential of each Jordan block separately.

Matrix Exponentials, VII

Proposition (Exponential of Jordan Block)

We have $e^{Jx} = \begin{bmatrix} e^{\lambda x} & xe^{\lambda x} & \frac{x^2}{2}e^{\lambda x} & \cdots & \frac{x^{d-1}}{(d-1)!}e^{\lambda x} \\ & e^{\lambda x} & xe^{\lambda x} & \ddots & \vdots \\ & & \ddots & \ddots & \frac{x^2}{2}e^{\lambda x} \\ & & & e^{\lambda x} & xe^{\lambda x} \\ & & & & e^{\lambda x} \end{bmatrix}$*, where* J *is the* $d \times d$ *Jordan block matrix with eigenvalue* λ*.*

Matrix Exponentials, VIII

Proof:

- Write $J = \lambda I + N$. As we showed earlier, N^d is the zero matrix, and $NI = IN$ since I is the identity matrix.
- Applying the binomial expansion yields $(Jx)^k = x^k(\lambda I + N)^k =$ $x^k \left[\lambda^k I + \binom{k}{1}\lambda^{k-1}N^1 + \cdots + \binom{k}{k-d}\lambda^{k-d}N^d + \cdots\right]$, but since N^d is the zero matrix, only the terms up through N^{d-1} are nonzero. (Note that we are using the fact that $IN = NI$, since the binomial theorem does not hold for general matrices.)
- It is then a straightforward (if somewhat lengthy) computation to plug these expressions into the infinite sum defining e^{Jx} and evaluate the infinite sum to obtain the stated result.

Matrix Exponentials, IX

Example: Solve the initial-value problem

$$\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{y}, \text{ where } \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 3 \end{bmatrix}.$$

Matrix Exponentials, IX

Example: Solve the initial-value problem

$$\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{y}, \text{ where } \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 3 \end{bmatrix}.$$

- Observe that A is already in Jordan canonical form.
- Hence $e^{At} = \begin{bmatrix} e^{2t} & te^{2t} & t^2e^{2t}/2 & 0 \\ 0 & e^{2t} & te^{2t} & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^t \end{bmatrix}$, so the solution is

$$\mathbf{y}(t) = e^{At} \begin{bmatrix} 1 \\ 2 \\ -4 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} e^{2t} + 2te^{2t} + 2t^2e^{2t} \\ 2e^{2t} - 4te^{2t} \\ -4e^{2t} \\ 3e^t \end{bmatrix}}.$$

Summary

We discussed the basic terminology for systems of linear differential equations.

We outlined the eigenvalue method for solving systems of homogeneous linear differential equations with constant coefficients.

We defined matrix exponentials and showed how to use them to solve systems of homogeneous linear differential equations with constant coefficients.

Next lecture: Bilinear Forms