

Math 4571 (Advanced Linear Algebra)

Lecture #26

The Jordan Canonical Form:

- Chains of Generalized Eigenvectors
- Existence and Uniqueness of the Jordan Canonical Form
- Computing the Jordan Canonical Form

This material represents §4.3.2 from the course notes.

Chains, I

In the last lecture, we showed that if $T : V \rightarrow V$ is linear and all the eigenvalues of T lie in the scalar field F , then V has a basis of generalized eigenvectors of V .

We now refine this result to show that we can select this basis in such a way that the associated matrix is in Jordan canonical form.

Important Note: Like in the last lecture, the proofs of the results in this lecture are fairly technical, and it is NOT necessary to follow all of the details. The important part is to understand what the theorems say.

Chains, II

Let us again examine the situation where we have a basis $\beta = \{\mathbf{v}_{k-1}, \mathbf{v}_{k-2}, \dots, \mathbf{v}_1, \mathbf{v}_0\}$ of V such that $T : V \rightarrow V$ has

associated matrix $[T]_{\beta}^{\beta} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, a Jordan block matrix.

- Then $T\mathbf{v}_{k-1} = \lambda\mathbf{v}_{k-1}$ and $T(\mathbf{v}_i) = \lambda\mathbf{v}_i + \mathbf{v}_{i+1}$ for each $0 \leq i \leq k-2$.
- Rearranging, we see that $(T - \lambda I)\mathbf{v}_{k-1} = \mathbf{0}$ and $(T - \lambda I)\mathbf{v}_i = \mathbf{v}_{i+1}$ for each $0 \leq i \leq k-2$.
- We can see that \mathbf{v}_0 is a generalized λ -eigenvector of T and that $\mathbf{v}_i = (T - \lambda I)^i \mathbf{v}_0$ for each $0 \leq i \leq k-1$.
- In other words, the basis β is composed of a “chain” of generalized eigenvectors obtained by successively applying the operator $T - \lambda I$ to a particular generalized eigenvector \mathbf{v}_0 .

Chains, III

Definition

Suppose $T : V \rightarrow V$ is linear and \mathbf{v} is a generalized λ -eigenvector of T such that $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$ and k is minimal. The list $\{\mathbf{v}_{k-1}, \mathbf{v}_{k-2}, \dots, \mathbf{v}_1, \mathbf{v}_0\}$, where $\mathbf{v}_i = (T - \lambda I)^i \mathbf{v}$ for each $0 \leq i \leq k - 1$, is called a chain of generalized eigenvectors.

It is not too hard to see (by running our previous logic backwards) that if we have a chain of generalized eigenvectors, then the corresponding associated matrix is a Jordan-block matrix.

Our goal is to prove that there exists a basis for the generalized λ -eigenspace consisting of chains of generalized eigenvectors: by applying this to each generalized eigenspace, we obtain a Jordan canonical form for T .

Chains, IV

A simple way to construct chains of generalized eigenvectors is simply to find a generalized eigenvector and then repeatedly apply $T - \lambda I$ to it.

However, this approach is rather haphazard, and it is not clear how to show that we can construct a basis consisting of chains.

Chains, V

Example: If $A = \begin{bmatrix} -1 & 2 & -2 & 1 \\ -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$, find a chain of generalized 1-eigenvectors for A having length 3.

Chains, \mathbf{v}

Example: If $A = \begin{bmatrix} -1 & 2 & -2 & 1 \\ -1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$, find a chain of generalized 1-eigenvectors for A having length 3.

- We compute $\det(tI - A) = t(t - 1)^3$. Thus, the eigenvalues of A are $\lambda = 0, 1, 1, 1$.
- By our theorems, the generalized 1-eigenspace is 3-dimensional and equal to the nullspace of $(A - I)^3$. Row-reducing gives a basis $\{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$.
- The vector $(1, 1, 0, 0)$ is an eigenvector of A , so it only produces a chain of length 1.
- But if we choose $\mathbf{v} = (0, 1, 1, 0)$, then $(A - I)\mathbf{v} = (0, 0, 0, -1)$ and $(A - I)^2\mathbf{v} = (-1, -1, -1, 0)$, so these three vectors form a chain of length 3.

Chains, VI

Example: If $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, show that the generalized 1-eigenspace of A cannot be spanned by a single chain.

Chains, VI

Example: If $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, show that the generalized

1-eigenspace of A cannot be spanned by a single chain.

- Notice that the eigenvalues of A are $\lambda = 1, 1, 1, 1$, since A is upper-triangular (in fact, it is in Jordan canonical form).
- However, observe that $(A - I)^2$ is the zero matrix.
- Hence, $(A - I)^2 \mathbf{v} = \mathbf{0}$ for any generalized 1-eigenvector \mathbf{v} .
- In particular, any chain of generalized 1-eigenvectors has length at most 2. Since the dimension of the generalized 1-eigenspace is 4, it cannot be spanned by a single chain.
- It is not hard to see, however, that it is spanned by the two chains $\{(0, 1, 0, 0), (1, 0, 0, 0)\}$ and $\{(0, 0, 0, 1), (0, 0, 1, 0)\}$.

Jordan Basis, I

If we are looking to construct a chain of generalized eigenvectors more systematically, we could instead run the construction in the opposite direction, by starting with a collection of eigenvectors and trying to find generalized eigenvectors mapped to them by $T - \lambda I$. By refining this idea appropriately, we can construct a basis for V consisting of chains of generalized eigenvectors:

Theorem (Existence of Jordan Basis)

If V is finite-dimensional, $T : V \rightarrow V$ is linear, and all eigenvalues of T lie in the scalar field of V , then V has a basis consisting of chains of generalized eigenvectors of T .

Jordan Basis, II

Proof:

- It suffices to show that each eigenspace has a basis consisting of chains of generalized eigenvectors, since (as we already showed) the union of bases for the generalized eigenspaces will be a basis for V .
- So suppose λ is an eigenvalue of T , let W be the generalized λ -eigenspace of V , with $\dim(W) = d$.
- Also, take $S : W \rightarrow W$ to be the map $S = T - \lambda I$, and note (as we showed) that S^d is the zero transformation on W .
- We must then prove that there exist vectors $\mathbf{w}_1, \dots, \mathbf{w}_k$ and integers a_1, \dots, a_k such that $S^{a_i}(\mathbf{w}_i) = \mathbf{0}$ and the set $\{\mathbf{w}_1, S\mathbf{w}_1, \dots, S^{a_1-1}\mathbf{w}_1, \mathbf{w}_2, S\mathbf{w}_2, \dots, S^{a_2-1}\mathbf{w}_2, \dots, \mathbf{w}_k, \dots, S^{a_k-1}\mathbf{w}_k\}$ is a basis of W .

Jordan Basis, III

Proof (continued):

- We will show this result by (strong) induction on d . If $d = 1$ then the result is trivial, since then S is the zero transformation so we can take $a_1 = 1$ and \mathbf{w}_1 to be any nonzero vector in W .
- Now assume $d \geq 2$ and that the result holds for all spaces of dimension less than d .
- Since $S : W \rightarrow W$ is not one-to-one (else it would be an isomorphism, but then S^d could not be zero) $W' = \text{im}(S)$ has dimension strictly less than $d = \dim(W)$.
- If W' is the zero space, then we can take $a_1 = \cdots = a_k = 1$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ to be any basis of W .

Jordan Basis, IV

Proof (more continued):

- Otherwise, if $W' = \text{im}(S)$ is not zero, then by the inductive hypothesis, there exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ and integers a_1, \dots, a_k such that $S^{a_i}(\mathbf{v}_i) = \mathbf{0}$ and the set $\beta' = \{\mathbf{v}_1, \dots, S^{a_1-1}\mathbf{v}_1, \dots, \mathbf{v}_k, \dots, S^{a_k-1}\mathbf{v}_k\}$ is a basis of W' .
- Now, since each \mathbf{v}_i is in $W' = \text{im}(S)$, by definition there exists a vector \mathbf{w}_i in W with $S\mathbf{w}_i = \mathbf{v}_i$. (In other words, can “extend” each of the chains for W' to obtain chains for W .)
- Furthermore, note that $\{S^{a_1-1}\mathbf{v}_1, \dots, S^{a_k-1}\mathbf{v}_k\}$ are linearly independent vectors in $\ker(S)$, so we can extend that set to obtain a basis $\gamma = \{S^{a_1-1}\mathbf{v}_1, \dots, S^{a_k-1}\mathbf{v}_k, \mathbf{z}_1, \dots, \mathbf{z}_s\}$ of $\ker(S)$.

Jordan Basis, V

Proof (yet more continued):

- We claim that the set $\beta = \{\mathbf{w}_1, \dots, S^{a_1}\mathbf{w}_1, \dots, \mathbf{v}_k, \dots, S^{a_k}\mathbf{w}_k, \mathbf{z}_1, \dots, \mathbf{z}_s\}$ is the desired basis for W . It clearly has the proper form, since $S\mathbf{z}_i = \mathbf{0}$ for each i , and the total number of vectors is $a_1 + \dots + a_k + s + k$.
- Furthermore, since $\{\mathbf{v}_1, \dots, S^{a_1-1}\mathbf{v}_1, \dots, \mathbf{v}_k, \dots, S^{a_k-1}\mathbf{v}_k\}$ is a basis of W' , $\dim(\text{im } T) = a_1 + \dots + a_k$, and since $\{S^{a_1-1}\mathbf{v}_1, \dots, S^{a_k-1}\mathbf{v}_k, \mathbf{z}_1, \dots, \mathbf{z}_s\}$ is a basis of $\ker(T)$, we see $\dim(\ker T) = s + k$.
- Then $\dim(W) = \dim(\ker T) + \dim(\text{im } T) = a_1 + \dots + a_k + s + k$, and so we see that the set β contains the proper number of vectors.

Jordan Basis, VI

Proof (even yet still more continued):

- It remains to verify that β is linearly independent. So suppose that $c_{1,1}\mathbf{w}_1 + \cdots + c_{k,a_k}S^{a_k-1}\mathbf{w}_k + b_1\mathbf{z}_1 + \cdots + b_s\mathbf{z}_s = \mathbf{0}$.
- Since $S^m\mathbf{w}_i = S^{m-1}\mathbf{v}_i$, applying S to both sides yields $c_{1,1}\mathbf{v}_1 + \cdots + c_{k,a_k-1}S^{a_k-1}\mathbf{v}_k = \mathbf{0}$, so since β' is linearly independent, all coefficients must be zero.
- The original dependence then reduces to $c_{1,a_1}S^{a_1}\mathbf{w}_1 + \cdots + c_{k,a_k}\mathbf{w}_k + b_1\mathbf{z}_1 + \cdots + b_s\mathbf{z}_s = \mathbf{0}$, but since γ is linearly independent, all coefficients must be zero. Thus, β is linearly independent and therefore a basis for W .
- Hence, each generalized eigenspace has a basis of the required form, and so by taking the union of these bases, we (at last!) obtain such a basis for V , as required.

Jordan Basis, VII

By applying the theorem we just proved, we can establish the existence of the Jordan form, which also turns out to be essentially unique:

Theorem (Jordan Canonical Form)

If V is finite-dimensional, $T : V \rightarrow V$ is linear, and all eigenvalues of T lie in the scalar field of V , then there exists a basis β of V such that $[T]_{\beta}^{\beta}$ is a matrix in Jordan canonical form. Furthermore, the Jordan canonical form is unique up to rearrangement of the Jordan blocks.

Jordan Basis, VIII

Proof:

- The existence of the Jordan basis follows from the Theorem we just proved: if we take β to be a basis consisting of chains of generalized eigenvectors, then as we have shown previously, $[T]_{\beta}^{\beta}$ is in Jordan canonical form.
- For the uniqueness, we claim that the number of Jordan blocks of eigenvalue λ having size at least d is equal to $\dim(\ker(T - \lambda I)^{d-1}) - \dim(\ker(T - \lambda I)^d)$.
- Since this quantity depends only on T (and not on the particular choice of basis) and completely determines the exact number of each type of Jordan block, the number of Jordan blocks of each size and eigenvalue must be the same in any Jordan canonical form.

Jordan Basis, IX

Proof (continued):

- To see this, let $S = T - \lambda I$ and take $\{\mathbf{w}_1, S\mathbf{w}_1, \dots, S^{a_1-1}\mathbf{w}_1, \mathbf{w}_2, S\mathbf{w}_2, \dots, S^{a_2-1}\mathbf{w}_2, \dots, \mathbf{w}_k, \dots, S^{a_k-1}\mathbf{w}_k\}$ to be a Jordan basis for the generalized λ -eigenspace: the sizes of the Jordan blocks are then $a_1 \leq a_2 \leq \dots \leq a_k$.
- It is not hard to see that a basis for the kernel of S^d is given by $\{S^{a_i-d}\mathbf{w}_i, \dots, S^{a_i-1}\mathbf{w}_i, \dots, S^{a_i-d}\mathbf{w}_k, \dots, S^{a_k-1}\mathbf{w}_k\}$, where i is the smallest value such that $d \leq a_i$.
- We can see that in extending the basis of $\ker(S^{d-1})$ to a basis of $\ker(S^d)$, we adjoin the additional vectors $\{S^{a_i-d}\mathbf{w}_i, S^{a_{i+1}-d}\mathbf{w}_{i+1}, \dots, S^{a_k-d}\mathbf{w}_k\}$, and the number of such vectors is precisely the number of a_i that are at least d .
- Thus, $\dim(\ker S^{d-1}) - \dim(\ker S^d)$ is the number of Jordan blocks of size at least d , as claimed.

Jordan Basis, X

In addition to proving the existence of the Jordan canonical form, the Theorem also gives us a method for computing it explicitly: all we need to do is find the dimensions of $\ker(T - \lambda I)$, $\ker(T - \lambda I)^2$, \dots , $\ker(T - \lambda I)^d$ where d is the multiplicity of the eigenvalue λ , and then find the numbers of each type of Jordan block.

- From the analysis above, the number of $d \times d$ Jordan blocks with eigenvalue λ is equal to $-\dim(\ker(T - \lambda I)^{d+1}) + 2\dim(\ker(T - \lambda I)^d) - \dim(\ker(T - \lambda I)^{d-1})$, which, by the nullity-rank theorem, is also equal to $\text{rank}((T - \lambda I)^{d+1}) - 2\text{rank}((T - \lambda I)^d) + \text{rank}((T - \lambda I)^{d-1})$.
- When actually working with the Jordan form J of a particular matrix A , one also wants to know the conjugating matrix Q with $A = Q^{-1}JQ$. We can take the columns of Q to be chains of generalized eigenvectors, but actually computing these chains is more difficult.

Examples, I

Example: Suppose the 3×3 matrix A has characteristic polynomial $p(t) = (t - 1)^2(t - 3)$. Find all possible Jordan canonical forms of A , up to equivalence.

Examples, I

Example: Suppose the 3×3 matrix A has characteristic polynomial $p(t) = (t - 1)^2(t - 3)$. Find all possible Jordan canonical forms of A , up to equivalence.

- Since 3 is a non-repeated eigenvalue, there can only be a Jordan block of size 1 associated to it.
- Since 1 is a double eigenvalue, the sum of the sizes of the corresponding Jordan blocks is 2. This gives two possibilities: either two blocks of size 1, or one block of size 2.
- Thus, up to equivalence, there are two possible Jordan

canonical forms: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Examples, II

Example: Suppose the 4×4 matrix A has characteristic polynomial $p(t) = (t - 2)^4$. Find all possible Jordan canonical forms of A , up to equivalence.

Examples, II

Example: Suppose the 4×4 matrix A has characteristic polynomial $p(t) = (t - 2)^4$. Find all possible Jordan canonical forms of A , up to equivalence.

- Since 2 is a repeated eigenvalue, the sum of the sizes of the corresponding Jordan blocks is 4. This gives five possibilities for the possible block sizes: 4, 3-1, 2-2, 2-1-1, and 1-1-1-1.
- This yields five possible Jordan canonical forms:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Examples, III

Example: Find the Jordan form of $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -4 & 3 & 1 & 3 \\ -5 & 3 & 2 & 4 \\ 3 & -1 & -1 & -1 \end{bmatrix}$.

- Since $\det(tI - A) = (t - 1)^4$, all of the Jordan blocks have eigenvalue 1. To find the sizes, we compute the ranks of the matrices $(A - I)^k$ for $k \geq 1$.
- Row-reducing shows $\text{rank}(A - I) = 2$, while $(A - I)^2$ is the zero matrix so $\text{rank}(A - I)^k = 0$ for $k \geq 2$.
- Thus, the number of 1×1 Jordan blocks is $\text{rank}(A - I)^2 - 2\text{rank}(A - I)^1 + \text{rank}(A - I)^0 = 0 - 2 \cdot 2 + 4 = 0$, and the number of 2×2 Jordan blocks is $\text{rank}(A - I)^3 - 2\text{rank}(A - I)^2 + \text{rank}(A - I)^1 = 0 - 2 \cdot 0 + 2 = 2$.

Examples, IV

- Thus, there are 2 blocks of size 2 with eigenvalue 1, and no blocks of other sizes or other eigenvalues.
- This means that the Jordan canonical form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Examples, V

Example: Find the Jordan form of $A = \begin{bmatrix} 0 & -1 & 3 & 2 \\ 1 & 0 & -2 & 0 \\ -1 & 0 & 3 & 1 \\ 2 & -1 & -3 & 0 \end{bmatrix}$.

- We compute $\det(tI - A) = t(t - 1)^3$, so the eigenvalues of A are $\lambda = 0, 1, 1, 1$. Since 0 is a non-repeated eigenvalue, there can only be a Jordan block of size 1 associated to it.
- To find the Jordan blocks with eigenvalue 1, we compute the ranks of the powers of $A - I$.

Examples, VI

- By row-reducing, we see $\text{rank}(A - I) = 3$, $\text{rank}(A - I)^2 = 2$, and $\text{rank}(A - I)^3 = 1$.
- Therefore, for $\lambda = 1$, we see that there are $2 - 2 \cdot 3 + 4 = 0$ blocks of size 1, $1 - 2 \cdot 2 + 3 = 0$ blocks of size 2, and $1 - 2 \cdot 1 + 2 = 1$ block of size 3.
- This means there is a Jordan 1-block of size 3 (along with the Jordan 0-block of size 1), and so the Jordan canonical form is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Summary

We discussed Jordan-block matrices and chains of generalized eigenvectors, and proved that every generalized eigenspace has a basis consisting of chains of generalized eigenvectors.

We established that if all the eigenvalues lie in the scalar field, then any linear transformation has a basis with respect to which its associated matrix is in Jordan canonical form, and that the Jordan canonical form is unique up to rearrangement.

We gave examples of how to compute the Jordan canonical form of a matrix.

Next lecture: Applications of the Jordan Canonical Form