

Math 4571 (Advanced Linear Algebra)

Lecture #25

Generalized Eigenvectors:

- Jordan-Block Matrices and the Jordan Canonical Form
- Generalized Eigenvectors
- Generalized Eigenspaces and the Spectral Decomposition

This material represents §4.3.1 from the course notes.

Jordan Canonical Form, I

In the last lecture, we discussed diagonalizability and showed that there exist matrices that are not conjugate to any diagonal matrix. For computational purposes, however, we might still like to know what the simplest form to which a non-diagonalizable matrix is similar.

The answer is given by what is called the Jordan canonical form, which we now describe.

Important Note: The proofs of the results in this lecture are fairly technical, and it is NOT necessary to follow all of the details. The important part is to understand what the theorems say.

Jordan Canonical Form, II

Definition

The $n \times n$ Jordan block with eigenvalue λ is the $n \times n$ matrix J having λ s on the diagonal, 1s directly above the diagonal, and zeroes elsewhere.

Here are the general Jordan block matrices of sizes 2, 3, 4, and 5:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Jordan Canonical Form, III

Definition

A matrix is in Jordan canonical form if it is a block-diagonal matrix

$$\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix}, \text{ where each } J_1, \dots, J_k \text{ is a Jordan block}$$

matrix (possibly with different eigenvalues and different sizes).

Example:

- The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is in Jordan canonical form, with $J_1 = [2]$, $J_2 = [3]$, $J_3 = [4]$.
- Indeed, any diagonal matrix is in Jordan canonical form.

Jordan Canonical Form, IV

Examples (continued):

- The matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is in Jordan canonical form, with

$$J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } J_2 = [3].$$

- The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in Jordan canonical form, with

$$J_1 = [1], J_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, J_3 = [1].$$

- Any single Jordan block matrix J is in Jordan canonical form, with $J_1 = J$.

Jordan Canonical Form, V

Examples (continued more):

- The matrix $\begin{bmatrix} \pi & 1 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 1 \\ 0 & 0 & 0 & \pi \end{bmatrix}$ is in Jordan canonical form,

$$\text{with } J_1 = J_2 = \begin{bmatrix} \pi & 1 \\ 0 & \pi \end{bmatrix}.$$

- The matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in Jordan canonical form, with

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } J_2 = [0].$$

Jordan Canonical Form, VI

Our goal is to prove that every matrix is similar to a Jordan canonical form and to give a procedure for computing the Jordan canonical form of a matrix.

Ultimately, a non-diagonalizable linear transformation (or matrix) fails to have enough eigenvectors for us to construct a diagonal basis. By generalizing the definition of eigenvector, we can fill in these “missing” basis entries.

If we then construct bases of these generalized eigenspaces in a particularly good way, the corresponding associated matrix will be in Jordan canonical form.

Jordan Canonical Form, VII

To motivate our discussion, suppose that there is a basis $\beta = \{\mathbf{v}_{k-1}, \mathbf{v}_{k-2}, \dots, \mathbf{v}_1, \mathbf{v}_0\}$ of V such that $T : V \rightarrow V$ has

associated matrix $[T]_{\beta}^{\beta} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, a Jordan block matrix.

- Then $T\mathbf{v}_{k-1} = \lambda\mathbf{v}_{k-1}$ and $T(\mathbf{v}_i) = \lambda\mathbf{v}_i + \mathbf{v}_{i+1}$ for each $0 \leq i \leq k-2$.
- Rearranging, we see that $(T - \lambda I)\mathbf{v}_{k-1} = \mathbf{0}$ and $(T - \lambda I)\mathbf{v}_i = \mathbf{v}_{i+1}$ for each $0 \leq i \leq k-2$.
- By an easy induction, $(T - \lambda I)^{k-i}\mathbf{v}_i = \mathbf{0}$ for each $0 \leq i \leq k$.
- What this means is: instead of having the elements in the basis be eigenvectors (elements in the kernel of $T - \lambda I$), they are instead elements in the kernel of some power of $T - \lambda I$.

Generalized Eigenvectors, I

This discussion motivates our definition of generalized eigenvectors:

Definition

For a linear operator $T : V \rightarrow V$, a nonzero vector \mathbf{v} satisfying $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$ for some positive integer k and some scalar λ is called a generalized eigenvector of T .

We take the same definition for matrices: a generalized eigenvector for A is a nonzero vector \mathbf{v} with $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$ for some positive integer k and some scalar λ .

Generalized Eigenvectors, II

Obviously, every (regular) eigenvector is also a generalized eigenvector (simply take $k = 1$). But there can exist generalized eigenvectors that are not (regular) eigenvectors.

Example: Show that $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is a generalized 2-eigenvector for

$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ that is not a (regular) 2-eigenvector.

- We compute $(A - 2I)\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$, and since this is not zero, \mathbf{v} is not a 2-eigenvector.
- However, $(A - 2I)^2\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and so \mathbf{v} is a generalized 2-eigenvector, with $k = 2$.

Generalized Eigenvectors, III

Like the (regular) eigenvectors, the generalized λ -eigenvectors (together with the zero vector) also form a subspace.

Proposition (Generalized Eigenspaces)

For a linear operator $T : V \rightarrow V$, the set of vectors \mathbf{v} satisfying $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$ for some positive integer k is a subspace of V . This subspace is called the generalized λ -eigenspace of T .

Proof: We verify the subspace criterion.

- [S1]: Clearly, the zero vector satisfies the condition.
- [S2]: If \mathbf{v}_1 and \mathbf{v}_2 have $(T - \lambda I)^{k_1} \mathbf{v}_1 = \mathbf{0}$ and $(T - \lambda I)^{k_2} \mathbf{v}_2 = \mathbf{0}$, then $(T - \lambda I)^{\max(k_1, k_2)} (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0}$.
- [S3]: If $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$, then $(T - \lambda I)^k (c\mathbf{v}) = \mathbf{0}$ as well.

Generalized Eigenvectors, IV

Although it may seem that we have also generalized the idea of an eigenvalue, in fact generalized eigenvectors can only have their associated constant λ be an eigenvalue of T :

Proposition (Eigenvalues for Generalized Eigenvectors)

If $T : V \rightarrow V$ is a linear operator and \mathbf{v} is a nonzero vector satisfying $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$ for some positive integer k and some scalar λ , then λ is an eigenvalue of T . Furthermore, the eigenvalue associated to a generalized eigenvector is unique.

Generalized Eigenvectors, \mathbf{v}

Proof:

- Let k be the smallest positive integer for which $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$. Then by assumption, $\mathbf{w} = (T - \lambda I)^{k-1} \mathbf{v}$ is not the zero vector, but $(T - \lambda I)\mathbf{w} = \mathbf{0}$.
- Thus, \mathbf{w} is an eigenvector of T with corresponding eigenvalue λ . In particular this means λ is an eigenvalue of T .
- For uniqueness, we show that $T - \mu I$ (hence also $(T - \mu I)^n$) is one-to-one on the generalized λ -eigenspace for any $\mu \neq \lambda$.
- Suppose $\mathbf{v} \neq \mathbf{0}$ is in the generalized λ -eigenspace and $(T - \mu I)\mathbf{v} = \mathbf{0}$. Let k be minimal with $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$.
- Then $\mathbf{w} = (T - \lambda I)^{k-1} \mathbf{v}$ is nonzero and $(T - \lambda I)\mathbf{w} = \mathbf{0}$.
- Also, we see that $(T - \mu I)\mathbf{w} = (T - \mu I)(T - \lambda I)^{k-1} \mathbf{v} = (T - \lambda I)^{k-1} (T - \mu I)\mathbf{v} = (T - \lambda I)^{k-1} \mathbf{0} = \mathbf{0}$.
- Then \mathbf{w} would be a nonzero vector in both the λ -eigenspace and the μ -eigenspace, which is impossible.

Computing Generalized Eigenvectors, I

From the definition of generalized eigenvector alone, it may seem from the definition that the value k with $(\lambda I - T)^k \mathbf{v} = \mathbf{0}$ may be arbitrarily large. But in fact, it is always the case that we can choose $k \leq \dim(V)$ when V is finite-dimensional:

Theorem (Computing Generalized Eigenspaces)

If $T : V \rightarrow V$ is a linear operator and V is finite-dimensional, then the generalized λ -eigenspace of T is equal to $\ker(T - \lambda I)^{\dim(V)}$. In other words, if $(T - \lambda I)^k \mathbf{v} = \mathbf{0}$ for some positive integer k , then in fact $(T - \lambda I)^{\dim(V)} \mathbf{v} = \mathbf{0}$.

Computing Generalized Eigenvectors, II

Proof:

- Let $S = T - \lambda I$ and define $W_i = \ker(S^i)$ for each $i \geq 1$.
- Observe that $W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots$, since if $S^i \mathbf{v} = \mathbf{0}$ then $S^{i+k} \mathbf{v} = \mathbf{0}$ for each $k \geq 1$.
- We claim that if $W_i = W_{i+1}$, then all W_{i+k} are also equal to W_i for all $k \geq 1$: in other words, that if two consecutive terms in the sequence are equal, then all subsequent terms are equal.
- So suppose that $W_i = W_{i+1}$, and let \mathbf{v} be any vector in W_{i+2} . Then $\mathbf{0} = S^{i+2} \mathbf{v} = S^{i+1}(S\mathbf{v})$, meaning that $S\mathbf{v}$ is in $\ker(S^{i+1}) = W_{i+1} = W_i = \ker(S^i)$. Therefore, $S^i(S\mathbf{v}) = \mathbf{0}$, so that \mathbf{v} is actually in W_{i+1} .
- Therefore, $W_{i+2} = W_{i+1}$. By iterating this argument we conclude that $W_i = W_{i+1} = W_{i+2} = \cdots$ as claimed.

Computing Generalized Eigenvectors, III

Proof (continued):

- Returning to the original argument, observe that $\dim(W_1) \leq \dim(W_2) \leq \cdots \leq \dim(W_k) \leq \dim(V)$ for each $k \geq 1$.
- Thus, since the dimensions are all nonnegative integers, we must have $\dim(W_k) = \dim(W_{k+1})$ for some $k \leq \dim(V)$, as otherwise we would have $1 \leq \dim(W_1) < \dim(W_2) < \cdots < \dim(W_k)$, but this is not possible since $\dim(W_k)$ would then exceed $\dim(V)$. Then $W_k = W_{k+1} = W_{k+2} = \cdots = W_{\dim(V)} = W_{\dim(V)+1} = \cdots$.
- Finally, if \mathbf{v} is a generalized eigenvector, then it lies in some W_i , but since the sequence of subspaces W_i stabilizes at $W_{\dim(V)}$, we conclude that \mathbf{v} is contained in $W_{\dim(V)} = \ker(S^{\dim(V)}) = \ker(T - \lambda I)^{\dim(V)}$, as claimed.

Computing Generalized Eigenvectors, IV

The Theorem gives us a completely explicit way to find the vectors in a generalized eigenspace: first find all eigenvalues λ for T , and then compute the kernel of $(T - \lambda I)^{\dim(V)}$ for each eigenvalue λ .

- We will show later that it is not generally necessary to raise $T - \lambda I$ to the full power $\dim(V)$: in fact, it is sufficient to compute the kernel of $(T - \lambda I)^{d_i}$, where d_i is the multiplicity of λ as a root of the characteristic polynomial.
- The advantage of taking the power as $\dim(V)$, however, is that it does not depend on T or λ in any way.

Computing Generalized Eigenvectors, V

Example: Find the generalized eigenspaces of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

- The characteristic polynomial is $\det(tI - A) = (t - 1)^2(t - 2)$ so the eigenvalues are $\lambda = 1, 1, 2$.
- For the generalized 1-eigenspace, we must compute the nullspace of $(A - I)^3 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
- Upon row-reducing, we see that the generalized 1-eigenspace has dimension 2 and is spanned by $(0, 1, 0)$ and $(0, 0, 1)$.
- Note here that neither of the generalized 1-eigenvectors is a 1-eigenvector, and (in fact) the 1-eigenspace of A is only 1-dimensional. This means A is not diagonalizable.

Computing Generalized Eigenvectors, VI

Example: Find the generalized eigenspaces of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

- For the generalized 2-eigenspace, we must compute the

$$\text{nullspace of } (A - 2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 3 \\ 1 & -3 & -4 \end{bmatrix}.$$

- Upon row-reducing, we see that the generalized 2-eigenspace has dimension 1 and is spanned by $(1, -1, 1)$.

In the example, observe that V does not have a basis of eigenvectors of A since the 1-eigenspace is only 1-dimensional.

Nonetheless, V does possess a basis of *generalized* eigenvectors.

Basis of Generalized Eigenvectors, I

Our goal is now to prove that there always exists a basis of generalized eigenvectors for V . Like in our argument for (regular) eigenvectors, we first prove that generalized eigenvectors associated to different eigenvalues are linearly independent.

Theorem (Independence of Generalized Eigenvectors)

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are generalized eigenvectors of T associated to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent.

The proof is essentially the same as for regular eigenvectors, with a bit of added complexity.

Basis of Generalized Eigenvectors, II

Proof: Induction on n . Base case $n = 1$ is trivial.

- Now suppose $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$ for generalized eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with distinct $\lambda_1, \lambda_2, \dots, \lambda_n$.
- Suppose $(T - \lambda_1 I)^k \mathbf{v}_1 = \mathbf{0}$. Apply $(T - \lambda_1 I)^k$ to both sides:

$$\mathbf{0} = T(\mathbf{0}) = a_1(T - \lambda_1 I)^k \mathbf{v}_1 + \cdots + a_n(T - \lambda_1 I)^k \mathbf{v}_n = a_2(T - \lambda_1 I)^k \mathbf{v}_2 + \cdots + a_n(T - \lambda_1 I)^k \mathbf{v}_n.$$
- Notice that $(T - \lambda_1 I)^k \mathbf{v}_j$ lies in the generalized λ_j -eigenspace, for each j : if $(T - \lambda_j I)^a \mathbf{v}_j = \mathbf{0}$, then $(T - \lambda_j I)^a [(T - \lambda_1 I)^k \mathbf{v}_j] = (T - \lambda_1 I)^k [(T - \lambda_j I)^a \mathbf{v}_j] = (T - \lambda_1 I)^k \mathbf{0} = \mathbf{0}$.
- Hence by the inductive hypothesis, $a_j(T - \lambda_1 I)^k \mathbf{v}_j$ must be zero. If $a_j \neq 0$, then \mathbf{v}_j would be in both the generalized λ_j -eigenspace and the generalized λ_1 -eigenspace (impossible).
- Thus $a_j = 0$ for all $j \geq 2$. Then $a_1\mathbf{v}_1 = \mathbf{0}$ so $a_1 = 0$ as well, so the \mathbf{v}_j are linearly independent.

Basis of Generalized Eigenvectors, III

Next, we would like to compute the exact dimensions of the generalized eigenspaces.

To do this, we will first establish a result regarding associated matrices that will make the calculations easier:

Theorem (Upper-Triangular Associated Matrix)

Suppose $T : V \rightarrow V$ is a linear operator on a finite-dimensional vector space such that the scalar field of V contains all eigenvalues of T . If λ is an eigenvalue of T having multiplicity d , then there exists a basis β of V such that $[T]_{\beta}^{\beta}$ is upper-triangular and the last d entries on the diagonal are equal to λ .

Basis of Generalized Eigenvectors, IV

Proof:

- Induct on $n = \dim(V)$. Base case $n = 1$ is trivial.
- For the inductive step, let λ be any eigenvalue of T .
- Define $W = \text{im}(T - \lambda I)$: since λ is an eigenvalue of T , $\ker(T - \lambda I)$ has positive dimension, so $\dim(W) < \dim(V)$.
- We claim that the map $S : W \rightarrow V$ given by $S(\mathbf{w}) = T(\mathbf{w})$ has $\text{im}(S)$ contained in W , so that S will be a linear operator on W (to which we can then apply the inductive hypothesis).
- To see this, let $\mathbf{w} \in W$. Then $S(\mathbf{w}) = (T - \lambda I)\mathbf{w} + \lambda\mathbf{w}$, and both $(T - \lambda I)\mathbf{w}$ and $\lambda\mathbf{w}$ are in W : since W is a subspace, we conclude that $S(\mathbf{w})$ also lies in W .

Basis of Generalized Eigenvectors, V

Proof (continued):

- Now since S is a linear operator on $W = \text{im}(T - \lambda I)$, by hypothesis there exists a basis $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ for W such that the matrix $[S]_\gamma^\gamma$ is upper-triangular and all eigenvalues λ appear at the end.
- Extend γ to a basis $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V . We claim that $[T]_\beta^\beta$ also has the desired properties.
- The upper $k \times k$ portion of $[T]_\beta^\beta$ is the matrix $[S]_\gamma^\gamma$ which is upper-triangular by hypothesis. Furthermore, for each \mathbf{v}_i we can write $T(\mathbf{v}_i) = (T - \lambda I)\mathbf{v}_i + \lambda\mathbf{v}_i$, and $(T - \lambda I)\mathbf{v}_i$ is in W , hence is a linear combination of $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$.
- Thus, $[T]_\beta^\beta$ is upper-triangular, as claimed. Also, by construction, all of the eigenvalues λ will appear at the end of the diagonal.

Basis of Generalized Eigenvectors, VI

Proof (continued more):

- It remains to see that we actually end up with d entries λ on the diagonal when we are finished.
- To see this, first observe that the diagonal entries of $[T]_{\beta}^{\beta}$ are the eigenvalues of T (counted with multiplicity).
- Also observe that $\det(tI - T) = \det(tI - S) \cdot (t - \lambda)^{\dim(E_{\lambda})}$, where E_{λ} is the λ -eigenspace of T . Thus, all eigenvalues of S will also lie in the scalar field of V .
- Thus, if we have not yet obtained d diagonal entries equal to λ , then the operator S will still have λ as an eigenvalue, so we will generate at least one additional λ on the diagonal in the next step of the construction.
- Hence we must obtain exactly d entries of λ at the end of the diagonal, as claimed.

Basis of Generalized Eigenvectors, VII

We now exploit this particular matrix representation to compute the dimension of the generalized λ -eigenspace.

Theorem (Dimension of Generalized Eigenspace)

If V is finite-dimensional, $T : V \rightarrow V$ is linear, and λ is a scalar, then the dimension of the generalized λ -eigenspace is equal to the multiplicity d of λ as a root of the characteristic polynomial of T , and in fact the generalized λ -eigenspace is the kernel of $(T - \lambda I)^d$.

Example: Suppose the characteristic polynomial of T is $p(t) = t^3(t - 2)^2$. Then:

- The generalized 0-eigenspace has dimension 3 and is $\ker(T^3)$.
- The generalized 2-eigenspace has dimension 2 and is $\ker(T - 2I)^2$.

Basis of Generalized Eigenvectors, VIII

Proof:

- Suppose λ has multiplicity d as a root of the char. polynomial.
- Apply the Theorem just proven to select a basis β for which $[T]_{\beta}^{\beta}$ is upper-triangular and has the last d diagonal entries equal to λ . (The remaining diagonal entries are the other eigenvalues of T , which by hypothesis are not equal to λ .)
- Then, for $B = A - \lambda I$, we see that $B = \begin{bmatrix} D & * \\ 0 & U \end{bmatrix}$, where D is upper-triangular with nonzero entries on the diagonal, U is a $d \times d$ upper-triangular matrix with zeroes on the diagonal, and $*$ is some matrix whose entries are irrelevant.

Basis of Generalized Eigenvectors, IX

Proof (continued):

- If $B = \begin{bmatrix} D & * \\ 0 & U \end{bmatrix}$, then $B^{\dim(V)} = \begin{bmatrix} D^{\dim(V)} & * \\ 0 & U^{\dim(V)} \end{bmatrix}$.
- By a straightforward induction argument, the power U^k has zeroes in the k rows above the diagonal.
- Thus, U^d is the zero matrix, so $U^{\dim(V)}$ is also the zero matrix, since $d \leq \dim(V)$.
- The generalized λ -eigenspace then has dimension equal to the nullity of $(A - \lambda I)^{\dim(V)} = B^{\dim(V)}$, but since $D^{\dim(V)}$ is upper-triangular with nonzero entries on the diagonal, we see that the nullity of $B^{\dim(V)}$ is exactly d .
- Finally, the statement that the generalized λ -eigenspace is the kernel of $(T - \lambda I)^d$ follows from the observation that U^d is actually the zero matrix.

Basis of Generalized Eigenvectors, X

Example: Find the dimensions of the generalized eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \text{ and then verify the result by finding a}$$

basis for each generalized eigenspace. Also, decide whether or not A is diagonalizable.

- Some computation produces $\det(tI - A) = t^3(t - 1)$. Thus, the eigenvalues of A are $\lambda = 0, 0, 0, 1$.
- By the Theorem, the dimension of the generalized 0-eigenspace is 3 and the dimension of the generalized 1-eigenspace is 1.

Basis of Generalized Eigenvectors, XI

Example (continued):

- For the generalized 0-eigenspace, the nullspace of

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \text{ has basis } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- For the generalized 1-eigenspace, the nullspace of

$$I - A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ has basis vector } \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

- The matrix A is not diagonalizable because there is not a basis of (regular) eigenvectors, as the 0-eigenspace only has dimension 1 (it is spanned by $(1, 0, 0, 0)$, as can be seen by row-reducing A).

Basis of Generalized Eigenvectors, XII

At last, we can show that any finite-dimensional (complex) vector space has a basis of generalized eigenvectors:

Theorem (Spectral Decomposition)

If V is finite-dimensional, $T : V \rightarrow V$ is linear, and all eigenvalues of T lie in the scalar field of V , then V has a basis of generalized eigenvectors of T .

The structure of this argument is essentially the same as in the characterization of diagonalizable transformations: we show that the union of the bases for each generalized eigenspace gives a basis for V .

Basis of Generalized Eigenvectors, XIII: Triskaidekaphobia

Proof:

- Suppose the eigenvalues of T are λ_i with respective multiplicities d_i as roots of the characteristic polynomial, and let $\beta_i = \{\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,d_i}\}$ be a basis for the generalized λ_i -eigenspace for each $1 \leq i \leq k$.
- We claim that $\beta = \beta_1 \cup \dots \cup \beta_k$ is a basis for V .
- By the previous theorem, the number of elements in β_i is d_i ; then β contains $\sum_i d_i = \dim(V)$ vectors, so to show β is a basis it suffices to prove that β is linearly independent.

Basis of Generalized Eigenvectors, XIV: Really, More?

Proof (continued):

- So suppose we have a dependence $a_{1,1}\mathbf{v}_{1,1} + \cdots + a_{k,j}\mathbf{v}_{k,j} = \mathbf{0}$. Let $\mathbf{w}_i = \sum_j a_{i,j}\mathbf{v}_{i,j}$: observe that \mathbf{w}_i lies in the generalized λ_i -eigenspace and that $\mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k = \mathbf{0}$.
- If any of the \mathbf{w}_i were nonzero, then we would have a nontrivial linear dependence between generalized eigenvectors of T having distinct eigenvalues, which is impossible.
- Therefore, each $\mathbf{w}_i = \mathbf{0}$, meaning that $a_{i,1}\mathbf{v}_{i,1} + \cdots + a_{i,d_i}\mathbf{v}_{i,d_i} = \mathbf{0}$. But then since β_i is linearly independent, all of the coefficients $a_{i,j}$ must be zero.
- We conclude that β is linearly independent and is therefore a basis for V .

Summary

We discussed Jordan-block matrices and the Jordan canonical form, as motivation for generalized eigenvectors.

We defined generalized eigenvectors and established some of their basic properties.

We proved that the dimension of the generalized λ -eigenspace is the multiplicity of λ as a root of the characteristic polynomial.

We showed that if all eigenvalues of T lie in the scalar field of V , then V has a basis of generalized eigenvectors.

Next lecture: The Jordan Canonical Form