E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 9, due Thu Apr 2nd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) If A is a diagonalizable $n \times n$ matrix, then so is $A + I_n$.

   (b) If A is a non-diagonalizable matrix and $p(t)$ is the characteristic polynomial of A, then $p(A)$ is the zero matrix.

   (c) If a matrix is diagonalizable, then its Jordan canonical form is diagonal.

   (d) If the Jordan canonical form of a matrix is diagonal, then the matrix is diagonalizable.

   (e) For any scalar λ, the λ-eigenspace of T is a subspace of the generalized λ-eigenspace of T.

   (f) For any λ, a chain of generalized λ-eigenvectors is linearly independent.

   (g) There always exists some basis β such that the matrix $[T]_\beta^\beta$ is in Jordan canonical form.

   (h) Every matrix $A \in M_{n\times n}(\mathbb{C})$ has a Jordan canonical form.

   (i) Two matrices are similar if and only if they have equivalent Jordan canonical forms.

   (j) Two matrices are similar if and only if they have the same characteristic polynomials.

   (k) The matrix $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 0 & e \\ 5 & e & \pi \end{bmatrix}$ is diagonalizable.

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2. Solve the following problems:

   (a) Find a formula for the kth power of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$.

   (b) In the country of Diagonalizistan there are two cities, unimaginatively named City A and City B. Each year, $2/5$ of the residents of City A move to City B, and $2/3$ of the residents of City B move to City A; the remaining residents stay in their current city. If in year 0 the populations of Cities A and B are 2000 and 6000 residents respectively, find the populations of the two cities in year n as $n \to \infty$.

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3. Suppose the characteristic polynomial of the 5×5 matrix A is $p(t) = t^3(t-1)^2$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   (b) List all possible Jordan canonical forms of A up to equivalence.

   (c) If $\ker(A)$ and $\ker(A - I)$ are both 2-dimensional, what is the Jordan canonical form of A?

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4. Find the Jordan canonical form of each matrix A over $\mathbb{C}$. (Note that many of these matrices also appear in problem 2 of homework 8.)

   (a) $A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

   (e) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.

   (a) If $B = -\dfrac{1}{a_0}(A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]

   (b) Show that there exists a polynomial $q(x)$ of degree at most $n-1$ such that $A^{-1} = q(A)$.

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6. Suppose V is finite-dimensional and $T : V \to V$ is a projection, so that $T^2 = T$.

   (a) Show that the only possible eigenvalues of T are 0 and 1.

   (b) Show that T is diagonalizable. [Hint: Use the fact that $V = \ker(T) \oplus \operatorname{im}(T)$ as proven in problem 8(c) of homework 4.]

   (c) Show that, up to similarity, there are exactly $n+1$ different projection maps, each of which has a different rank.

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7. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) If J is a $d \times d$ Jordan block with eigenvalue λ, show that J is similar to its transpose. [Hint: Reverse the Jordan basis.]

   (b) If J is a matrix in Jordan canonical form, show that J is similar to its transpose.

   (c) Show that A is similar to its transpose.

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8. Suppose V is a (not necessarily finite-dimensional) complex inner product space, $T : V \to V$ is linear, and the adjoint T^* exists.

   (a) If $T^* = T$, show that $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$.

   (b) If $\langle T(\mathbf{x}), \mathbf{x}\rangle = \mathbf{0}$ for all $\mathbf{x} \in V$, show that T is the zero transformation. [Hint: Set $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and also $\mathbf{x} = \mathbf{a} + i\mathbf{b}$, and exploit the two equations.]

   (c) If $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$, show that $T^* = T$. [Hint: Compute $\langle (T^* - T)\mathbf{v}, \mathbf{v}\rangle$ and use part (b).]

   (d) If V is finite-dimensional and $\langle T(\mathbf{v}), \mathbf{v}\rangle$ is real for all $\mathbf{v} \in V$, conclude that T is diagonalizable.
