E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 8, due Thu Mar 26th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) For any $\lambda \in F$, the set of vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ is a subspace of V.

   (b) If $T(\mathbf{v}) = \lambda\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of T.

   (c) Every linear transformation on V has at least one eigenvector.

   (d) If V is finite-dimensional, every linear transformation on V has at least one eigenvector.

   (e) Any two eigenvectors are linearly independent.

   (f) The sum of two eigenvectors is also an eigenvector.

   (g) The sum of two eigenvalues is also an eigenvalue.

   (h) If two matrices are similar, then they have the same eigenvalues.

   (i) If two matrices are similar, then they have the same eigenvectors.

   (j) If two matrices have the same eigenvalues, then they are similar.

   (k) If $\dim(V) = n$, then T has at most n distinct eigenvalues in F.

   (l) If $\dim(V) = n$, then T has exactly n distinct eigenvalues in F.

   (m) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the 1-eigenspace of A has dimension 2.

   (n) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the only vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.

   (o) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.

   (p) If $\dim(V) = n$ and T has n distinct eigenvalues in F, then T is diagonalizable.

   (q) If $\dim(V) = n$ and T is diagonalizable, then T has n distinct eigenvalues in F.

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2. For each matrix A over each field F, (i) find all eigenvalues of A over F, (ii) find a basis for each eigenspace of A, and (iii) determine whether or not A is diagonalizable and if so find an invertible matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

   (a) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{R}$.

   (b) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{C}$.

   (c) The matrix $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Q}$.

   (d) The matrix $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.

   (e) The matrix $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$ over $\mathbb{R}$.

   (f) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

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3. For each operator $T : V \to V$ on each vector space V, (i) find all its eigenvalues and a basis for each eigenspace, and (ii) determine whether the operator is diagonalizable and if so, find a basis for which $[T]_\beta^\beta$ is diagonal:

   (a) The map $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $T(x, y) = (x + 4y, 3x + 5y)$.

   (b) The derivative operator $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $D(p) = p'$.

   (c) The transpose map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $T(M) = M^T$.

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4. Let F be a field and let L and R be the left shift and right shift operators on infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Find all of the eigenvalues and a basis for each eigenspace of L.

   (b) Find all of the eigenvalues and a basis for each eigenspace of R.

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5. Suppose A is a matrix with characteristic polynomial $p(t) = (t-1)^2(t-2)^4$.

   (a) What are the dimensions of A?

   (b) Find the eigenvalues of A and the possible dimensions of the corresponding eigenspaces.

   (c) Find the determinant and the trace of A.

   (d) If A is diagonalizable, find a diagonal matrix to which A is similar.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. The goal of this problem is to find the eigenvalues and eigenvectors of the $n \times n$ "all 1s" matrix over an arbitrary field F. So let $n \geq 2$ and let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

   (a) Show that the 0-eigenspace of A has dimension $n-1$ and find a basis for it.

   (b) If the characteristic of F does not divide n, find the remaining nonzero eigenvalue of A and a basis for the corresponding eigenspace, and show that A is diagonalizable. [Hint: Calculate the trace of A.]

   (c) If the characteristic of F does divide n, show that A is not diagonalizable. [Hint: Note that $\mathrm{char}(F)$ dividing n is the same as saying that $n = 0$ in F.]

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7. Suppose V is a vector space and $S, T : V \to V$ are linear operators on V.

   (a) If S and T commute (i.e., $ST = TS$), show that S maps each eigenspace of T into itself.

   (b) If $\mathbf{v}$ is an eigenvector of T, show that it is also an eigenvector of T^n for any positive integer n.

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8. Suppose V is an inner product space over the field F (where $F = \mathbb{R}$ or $\mathbb{C}$) and $T : V \to V$ is linear. We say T is a "distance-preserving" map on V if $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v}$ in V, and we say T is an "angle-preserving" map on V if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.

   (a) Prove that T is distance-preserving if and only if it is angle-preserving. [Hint: Use problem 3 from homework 7.]

   A map $T : V \to V$ satisfying the distance- and angle-preserving conditions is called a (linear) <u>isometry</u>.

   (b) Show that the transformations $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S(x, y, z) = (z, -x, y)$ and $T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z)$ are both isometries under the usual dot product.

   (c) Show that isometries are one-to-one.

   (d) Show that isometries preserve orthogonal and orthonormal sets.

   (e) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.

   (f) We say that a matrix $A \in M_{n \times n}(F)$ is <u>unitary</u> if $A^{-1} = A^*$. Show that the isometries of F^n (with its usual inner product) are precisely those maps given by left-multiplication by a unitary matrix.

   - <u>Remark</u>: Notice that $A \in M_{n \times n}(\mathbb{C})$ is unitary if and only if the columns of A are an orthonormal basis of $\mathbb{C}$. Thus, the result of part (f) can equivalently be thought of as saying that the distance-preserving maps on $\mathbb{C}^n$ (or $\mathbb{R}^n$) are simply changes of basis from one orthonormal basis (the columns of A) to another (the standard basis).
