E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 7, due Mon Mar 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V with scalar field F. Identify each of the following statements as true or false:

   (a) If V is finite-dimensional and W is any subspace of V, then $\dim(W) = \dim(W^\perp)$.

   (b) If V has an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $W = \text{span}(\mathbf{e}_1 + 2\mathbf{e}_3)$, then $W^\perp = \text{span}(\mathbf{e}_2, 2\mathbf{e}_1 - \mathbf{e}_3)$.

   (c) If $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V, then $\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n$ for any $\mathbf{w} \in W$.

   (d) If $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is an orthonormal basis of W, then $\mathbf{w} = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.

   (e) If V is finite-dimensional, $\mathbf{v} \in V$, and W is any subspace of V, the vector $\mathbf{w} \in W$ minimizing $||\mathbf{v} - \mathbf{w}||$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.

   (f) If $T : V \to V$ is linear, then the adjoint of T exists and is unique.

   (g) If $T : V \to V$ is linear and V is finite-dimensional, then the adjoint of T exists and is unique.

   (h) If $T : V \to F$ is linear and V is finite-dimensional, then there exists $\mathbf{w} \in V$ such that $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in V$.

   (i) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + 2T)^* = S^* + 2T^*$.

   (j) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + iT)^* = S^* + iT^*$.

   (k) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(ST)^* = S^* T^*$.

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2. Calculate the following things (assume any unspecified inner product is the standard one):

   (a) The angle between the vectors $\mathbf{v} = (3, 1, 2)$ and $\mathbf{w} = (1, -3, \sqrt{70})$ in $\mathbb{R}^3$.

   (b) The "angle" between the functions $f(x) = 1 + \sin x$ and $g(x) = 1 - \sin x$ in $C[0, 2\pi]$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$.

   (c) Write $\mathbf{v} = (9, 7, -8)$ as a linear combination of the orthogonal basis $(-1, 1, 2)$, $(2, 0, 1)$, $(1, 5, -2)$ of $\mathbb{R}^3$.

   (d) Write $\mathbf{v} = (-5, 5, -6)$ as a linear combination of the orthogonal basis $(i, -i, 0)$, $(1, 1, 2i)$, $(i, i, 1)$ of $\mathbb{C}^3$.

   (e) A basis for $W^\perp$, if $W = \text{span}[(1, 1, 1, 1), (2, 3, 4, 1)]$ inside $\mathbb{R}^4$.

   (f) A basis for $W^\perp$, if $W = \text{span}[(1, 1, 2i), (1, -i, 4)]$ inside $\mathbb{C}^3$. [Hint: Over $\mathbb{C}$, compute the complex conjugate of the nullspace.]

   (g) The orthogonal projection of $\mathbf{v} = (2, 0, 11)$ into $W = \text{span}[\frac{1}{3}(1, 2, 2), \frac{1}{3}(2, -2, 1)]$ inside $\mathbb{R}^3$, and also verify the relation $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2$.

   (h) The orthogonal projection of $\mathbf{v} = 1 + 2x^2$ into $W = \text{span}[x, x^2, x^3]$ with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$.

   (i) The least-squares solution to the inconsistent system $x + 3y = 9$, $3x + y = 5$, $x + y = 2$.

   (j) The line $y = mx + b$ that is the best model, using least squares, for the data points $\{(4, 7), (11, 21), (15, 29), (19, 35), (30, 49)\}$. (Give three decimal places.)

   (k) The quadratic $y = ax^2 + bx + c$ that is the best model, using least squares, for the data points $\{(-2, 22), (-1, 11), (0, 4), (1, 3), (2, 13)\}$. (Give three decimal places.)

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Let V be an inner product space with scalar field F. The goal of this problem is to prove the so-called "polarization identities".

   (a) If $F = \mathbb{R}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} ||\mathbf{v} + \mathbf{w}||^2 - \frac{1}{4} ||\mathbf{v} - \mathbf{w}||^2$.

   (b) If $F = \mathbb{C}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \left|\left|\mathbf{v} + i^k \mathbf{w}\right|\right|^2$.

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4. Let V be a finite-dimensional inner product space with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$.

   (a) For any $\mathbf{x} \in V$ show that $\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \cdots + \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n$.

   (b) For any $\mathbf{x}, \mathbf{y} \in V$ show that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{e}_1 \rangle \overline{\langle \mathbf{y}, \mathbf{e}_1 \rangle} + \langle \mathbf{x}, \mathbf{e}_2 \rangle \overline{\langle \mathbf{y}, \mathbf{e}_2 \rangle} + \cdots + \langle \mathbf{x}, \mathbf{e}_n \rangle \overline{\langle \mathbf{y}, \mathbf{e}_n \rangle}$.

   (c) For any $\mathbf{x} \in V$ show that $||\mathbf{x}||^2 = |\langle \mathbf{x}, \mathbf{e}_1 \rangle|^2 + |\langle \mathbf{x}, \mathbf{e}_2 \rangle|^2 + \cdots + |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2 = \sum_{i=1}^{n} |\langle \mathbf{x}, \mathbf{e}_i \rangle|^2$. What theorem from classical geometry does this generalize?

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5. Suppose $T : V \to W$ is a linear transformation and $\langle \cdot, \cdot \rangle_W$ is an inner product on W.

   (a) If T is one-to-one, show that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_V = \langle T(\mathbf{v}_1), T(\mathbf{v}_2) \rangle_W$ is an inner product on V.

   (b) Is the map defined in part (a) necessarily an inner product if the assumption that T is one-to-one is dropped? Explain why or why not.

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6. Let $V = C[0, 2\pi]$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,dx$. Also define $\varphi_0(x) = \frac{1}{\sqrt{2\pi}}$, and for positive integers k set $\varphi_{2k-1}(x) = \frac{1}{\sqrt{\pi}} \cos(kx)$ and $\varphi_{2k}(x) = \frac{1}{\sqrt{\pi}} \sin(kx)$.

   (a) Show that $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ is an orthonormal set in V.

   (b) Let $f(x) = x$. Find $||f||$ and $\langle f, \varphi_n \rangle$ for each $n \geq 0$.

   (c) With $f(x) = x$, assuming that $||f||^2 = \sum_{n=0}^{\infty} \langle f, \varphi_k \rangle^2$ (see problem 5(c) for why this is a reasonable statement), find the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   Remarks: The identity $||f||^2 = \sum_{n=0}^{\infty} \langle f, \varphi_k \rangle^2$ is known as Parseval's identity. The problem of computing the value of the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is known as the Basel problem. The correct value was (famously) first found by Euler, who evaluated the sum by decomposing the function $\frac{\sin(\pi x)}{\pi x}$ as the infinite product $\prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2})$ and then comparing the power series coefficients of both sides.

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7. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is a linear transformation possessing an adjoint T^*. We say T is Hermitian (or self-adjoint) if $T = T^*$, and that T is skew-Hermitian if $T = -T^*$.

   (a) Show that T is Hermitian if and only if iT is skew-Hermitian.

   (b) Show that $T + T^*$, T^*T, and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.

   (c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2.

   (d) Suppose T is Hermitian. Prove that $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number for any vector $\mathbf{v}$.
