E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 6, due Thu Feb 27th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Identify each of the following statements as true or false:
   (a) An inner product is linear in each of its components.
   (b) There is exactly one inner product on $\mathbb{R}^n$.
   (c) The Cauchy-Schwarz inequality holds in every inner product space.
   (d) The triangle inequality holds in real inner product spaces but not complex inner product spaces.
   (e) If $\langle \mathbf{v}, 2\mathbf{v} \rangle = 0$ then $\mathbf{v} = \mathbf{0}$.
   (f) The zero vector is orthogonal to itself.
   (g) If $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, 2\mathbf{w} \rangle$ then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
   (h) If V is a complex vector space, the vectors $\mathbf{v}$ and $i\mathbf{v}$ are always orthogonal.
   (i) Every finite-dimensional inner product space has an orthonormal basis.
   (j) An orthogonal set of vectors is linearly independent.
   (k) An orthonormal set of vectors is linearly independent.

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2. For each of the following pairings, determine (with justification) whether or not it is an inner product on the given vector space:
   (a) The pairing $\langle A, B \rangle = \operatorname{tr}(A + B)$ on $M_{2\times 2}(\mathbb{R})$.
   (b) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{R}^2$.
   (c) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{C}^2$.
   (d) The pairing $\langle (a, b), (c, d) \rangle = ac$ on $\mathbb{R}^2$.
   (e) The pairing $\langle f, g \rangle = \int_0^1 f'(x)g(x)\,dx$ on $C[0, 1]$.

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3. For each pair of vectors $\mathbf{v}, \mathbf{w}$ in the given inner product space, compute $\langle \mathbf{v}, \mathbf{w} \rangle$, $||\mathbf{v}||$, $||\mathbf{w}||$, and $||\mathbf{v} + \mathbf{w}||$, and verify the Cauchy-Schwarz and triangle inequalities for $\mathbf{v}$ and $\mathbf{w}$:
   (a) $\mathbf{v} = (1, 2, 2, 4)$ and $\mathbf{w} = (4, 1, 4, 4)$ in $\mathbb{R}^4$ with the standard inner product.
   (b) $\mathbf{v} = (i, -i, 1 + i)$ and $\mathbf{w} = (2 - i, 4, -2i)$ in $\mathbb{C}^3$ with the standard inner product.
   (c) $\mathbf{v} = 1 + t$ and $\mathbf{w} = 2 + t^2$ in $C[-1, 1]$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)\,dt$.
   (d) $\mathbf{v} = e^t$ and $\mathbf{w} = e^{2t}$ in $C[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

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4. For each list S of vectors in the given inner product space, apply Gram-Schmidt to calculate an orthogonal basis for $\operatorname{span}(S)$:
   (a) $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (1, 2, 0)$, $\mathbf{v}_3 = (1, 2, 3)$ in $\mathbb{R}^3$ under the standard dot product.
   (b) $\mathbf{v}_1 = (2, 4, -4)$, $\mathbf{v}_2 = (1, -1, 4)$, $\mathbf{v}_3 = (1, 1, 1)$ in $\mathbb{R}^3$ under the standard dot product.
   (c) $\mathbf{v}_1 = x$, $\mathbf{v}_2 = x^2$, $\mathbf{v}_3 = x^3$ in $C[-1, 1]$ under the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)\,dx$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Let V be an inner product space.

   (a) If $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on V, show that $\langle \cdot, \cdot \rangle_3 = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is also an inner product on V.

   (b) If $\langle \cdot, \cdot \rangle_1$ is an inner product on V and c is a positive real number, show that $\langle \cdot, \cdot \rangle_2 = c \langle \cdot, \cdot \rangle_1$ is also an inner product on V.

   (c) Does the collection of inner products on V form a vector space under the natural addition and scalar multiplication described above? Explain why or why not.

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6. Prove the following inequalities:

   (a) Prove that $(a_1 + a_2 + \cdots + a_n) \cdot (\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \geq n^2$ for any positive real numbers $a_1, a_2, \ldots, a_n$, with equality if and only if all of the a_i are equal.

   (b) If a, b, c, d are real numbers with $a^2 + b^2 + c^2 + d^2 \leq 5$, show that $a + 2b + 3c + 4d \leq 5\sqrt{6}$.

   (c) Prove Nesbitt's inequality: for any positive real numbers a, b, c it is true that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$. [Hint: Apply Cauchy-Schwarz to $(\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $(1/\sqrt{a+b}, 1/\sqrt{b+c}, 1/\sqrt{c+a})$ and then rearrange.]

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7. Let V be a finite-dimensional inner product space and W be a subspace of V.

   (a) Prove that $W \cap W^\perp = \{\mathbf{0}\}$ and deduce that $V = W \oplus W^\perp$. [Hint: Use the fact that $\dim(W) + \dim(W^\perp) = \dim(V)$.]

   (b) Let T be the "orthogonal projection" map from V to W, defined as the map taking $T(\mathbf{v}) = \mathbf{w}$ if $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ for $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$. Prove that T is a linear transformation that is a projection map (as defined in problem 8 of homework 4).

   (c) Show that $(W^\perp)^\perp = W$. [Hint: To show $(W^\perp)^\perp \subseteq W$, write $\mathbf{v} \in (W^\perp)^\perp$ as $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ and compute $\langle \mathbf{v}, \mathbf{w}^\perp \rangle$.]

   (d) If W_1 and W_2 are subspaces of V, prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. [Hint: The second statement follows from the first and part (c).]
