

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that the vector spaces U, V, W are finite-dimensional over the field F , the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that S, T are linear transformations. Identify each of the following statements as true or false:

- (a) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_{\beta}^{\gamma}$ is an element of $M_{m \times n}(F)$.
 - (b) If $[S]_{\alpha}^{\beta} = [T]_{\alpha}^{\beta}$ then $S = T$.
 - (c) If $[T]_{\alpha}^{\beta} = [T]_{\gamma}^{\delta}$ then $\alpha = \gamma$ and $\beta = \delta$.
 - (d) If $S : V \rightarrow W$ and $T : V \rightarrow W$ then $[S + T]_{\alpha}^{\beta} = [S]_{\alpha}^{\beta} + [T]_{\alpha}^{\beta}$.
 - (e) If $T : V \rightarrow W$ and $\mathbf{v} \in V$, then $[T]_{\alpha}^{\beta}[\mathbf{v}]_{\beta} = [T\mathbf{v}]_{\alpha}$.
 - (f) If $S : V \rightarrow W$ and $T : U \rightarrow V$, then $[ST]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$.
 - (g) If $T : V \rightarrow V$ has an inverse T^{-1} , then $[T^{-1}]_{\beta}^{\gamma} = ([T]_{\beta}^{\gamma})^{-1}$.
 - (h) If $T : V \rightarrow V$ has an inverse T^{-1} , then for any $\mathbf{v} \in V$, $[T^{-1}\mathbf{v}]_{\gamma} = ([T]_{\gamma}^{\beta})^{-1}[\mathbf{v}]_{\beta}$.
 - (i) If $T : V \rightarrow V$ and $[T]_{\beta}^{\gamma}$ is the identity matrix, then T must be the identity transformation.
 - (j) If $T : V \rightarrow V$ and $[T]_{\beta}^{\gamma}$ is the zero matrix, then T must be the zero transformation.
 - (k) The space $\mathcal{L}(V, W)$ of all linear transformations from V to W has dimension $\dim V \cdot \dim W$.
 - (l) If A is an $m \times n$ matrix of rank r , then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension r .
 - (m) If A is an $m \times n$ matrix and the linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then the rank of A is less than n .
 - (n) If A is an $n \times n$ matrix of rank n , then the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.
 - (o) If the columns of A are all scalar multiples of some vector \mathbf{v} , then $\text{rank}(A) \leq 1$.
 - (p) For any $T : V \rightarrow V$, there always exists an invertible matrix Q such that $[T]_{\beta}^{\beta} = Q^{-1}[T]_{\alpha}^{\alpha}Q$.
 - (q) For any $T : V \rightarrow V$, if $P = [I]_{\beta}^{\gamma}$, then it is true that $[T]_{\gamma}^{\gamma} = P[T]_{\beta}^{\beta}P^{-1}$.
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2. For each linear transformation T and given bases β and γ , find $[T]_{\beta}^{\gamma}$:

- (a) $T : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ given by $T(a, b) = \langle a - b, b - 2a, 3b \rangle$, with β and γ the standard bases.
 - (b) The trace map from $M_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ and $\gamma = \{1\}$.
 - (c) $T : \mathbb{Q}^4 \rightarrow P_4(\mathbb{Q})$ given by $T(a, b, c, d) = a + (a + b)x + (a + 3c)x^2 + (2a + d)x^3 + (b + 5c + d)x^4$, with β the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.
 - (d) $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ with $\beta = \gamma$ the standard basis.
 - (e) The projection map (see problem 8 of homework 4) on \mathbb{R}^3 that maps the vectors $\langle 1, 2, 1 \rangle$ and $\langle 0, -3, 1 \rangle$ to themselves and sends $\langle 1, 1, 1 \rangle$ to the zero vector, with $\beta = \gamma = \{ \langle 1, 2, 1 \rangle, \langle 0, -3, 1 \rangle, \langle 1, 1, 1 \rangle \}$.
 - (f) The same map as in part (e), but relative to the standard basis for \mathbb{R}^3 .
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3. Let $T : P_3(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ be given by $T(p) = x^2 p''(x)$.

(a) With the bases $\alpha = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$, find $[T]_{\alpha}^{\gamma}$.

(b) If $q(x) = 1 - x^2 + 2x^3$, compute $[q]_{\alpha}$ and $[T(q)]_{\gamma}$ and verify that $[T(q)]_{\gamma} = [T]_{\alpha}^{\gamma}[q]_{\alpha}$.

Notice that $T = SU$ where $U : P_3(\mathbb{R}) \rightarrow P_1(\mathbb{R})$ has $U(p) = p''(x)$ and $S : P_1(\mathbb{R}) \rightarrow P_4(\mathbb{R})$ has $S(p) = x^2 p(x)$.

(c) With $\beta = \{1, x\}$, compute the associated matrices $[S]_{\beta}^{\gamma}$, and $[U]_{\alpha}^{\beta}$ and then verify that $[T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma}[U]_{\alpha}^{\beta}$.

(d) Which of S , T , and U are onto? One-to-one? Isomorphisms?

4. Suppose $V = P_3(\mathbb{R})$, with standard basis $\beta = \{1, x, x^2, x^3\}$, and let $T : V \rightarrow V$ be the linear transformation with $T(1) = 1 - x + x^2 - x^3$, $T(x) = 2x - x^3$, and $T(x^2) = 3 + x - x^3$, and $T(x^3) = 1 - x^2$.

(a) Find $[T]_{\beta}^{\beta}$.

Now let γ be the ordered basis $\gamma = \{x^3, x^2, x + 1, x\}$.

(b) Find the change-of-basis matrix $Q = [I]_{\beta}^{\gamma}$ and its inverse.

(c) For $\mathbf{v} = 2 - x - 2x^2 + x^3$, compute $[\mathbf{v}]_{\beta}$, $[\mathbf{v}]_{\gamma}$, and verify that $[\mathbf{v}]_{\gamma} = Q[\mathbf{v}]_{\beta}$.

(d) Find $[T]_{\beta}^{\gamma}$, $[T]_{\gamma}^{\beta}$, and $[T]_{\gamma}^{\gamma}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose V and W are finite-dimensional vector spaces and $T : V \rightarrow W$ is linear.

(a) If β and γ are ordered bases of V and W respectively such that $[T]_{\beta}^{\gamma}$ is the identity matrix, show that T is an isomorphism.

(b) If T is an isomorphism, show that there exist ordered bases β and γ of V and W respectively such that $[T]_{\beta}^{\gamma}$ is the identity matrix.

6. Suppose that $T : V \rightarrow V$ is a linear transformation on a finite-dimensional vector space.

(a) If β and γ are two ordered bases of V , show that $\det([T]_{\beta}^{\beta}) = \det([T]_{\gamma}^{\gamma})$.

Per part (a), we define $\det(T)$ to be $\det([T]_{\beta}^{\beta})$ for any choice of ordered basis β .

(b) Show that T is an isomorphism if and only if $\det(T)$ is nonzero.

7. Let F be a field and $n \geq 2$ be an integer. Recall that we say two matrices A and B are similar if there exists an invertible matrix Q with $B = Q^{-1}AQ$.

(a) Show that if A and B are similar matrices in $M_{n \times n}(F)$, then $\det(A) = \det(B)$ and $\text{tr}(A) = \text{tr}(B)$. [Hint: You may use the fact that $\text{tr}(CD) = \text{tr}(DC)$.]

(b) Show that “being similar” is an equivalence relation on $M_{n \times n}(F)$.
