

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

---

**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:

- (a) The zero vector space has no basis.
  - (b) The set  $\{\mathbf{0}\}$  is a basis for the zero vector space.
  - (c) Every vector space has a finite basis.
  - (d) Every vector space has a unique basis.
  - (e) No vector space has a unique basis.
  - (f) Every subspace of a finite-dimensional vector space is finite-dimensional.
  - (g) Every subspace of an infinite-dimensional vector space is infinite-dimensional.
  - (h) If  $V = M_{m \times n}(F)$ , then  $\dim_F V = mn$ .
  - (i) If  $V = F[x]$ , then  $\dim_F V$  is undefined.
  - (j) If  $V = P_n(F)$ , then  $\dim_F V = n$ .
  - (k) If  $\dim(V) = 5$ , then there is a unique subspace of  $V$  of dimension 0.
  - (l) If  $\dim(V) = 5$ , then there is a unique subspace of  $V$  of dimension 1.
  - (m) If  $\dim(V) = 5$ , then there is a unique subspace of  $V$  of dimension 5.
  - (n) If  $\dim(V) = 5$ , then there exists a set of 5 vectors in  $V$  that span  $V$  but are not linearly independent.
  - (o) If  $V$  is infinite-dimensional, then any infinite linearly-independent subset is a basis.
- 

2. Find a basis for, and the dimension of, each of the following vector spaces:

- (a) The space of  $3 \times 3$  symmetric matrices over  $F = \mathbb{C}$ .
  - (b) The row space, column space, and nullspace of  $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$ .
  - (c) The vectors in  $\mathbb{Q}^5$  of the form  $\langle a, b, c, d, e \rangle$  with  $e = a + b$  and  $b = c = d$ , over  $\mathbb{Q}$ .
  - (d) The row space, column space, and nullspace of  $M = \begin{bmatrix} 1 & 3 & -2 & -6 & 8 \\ 2 & -1 & 2 & 8 & 1 \\ -1 & 1 & 1 & -3 & 3 \end{bmatrix}$ .
  - (e) The polynomials  $p(x)$  in  $P_4(\mathbb{R})$  such that  $p(i) = 0$ .
- 

3. Calculate the following:

- (a) If  $S = \{\langle 2, 1, -1 \rangle, \langle -1, 2, 3 \rangle, \langle -2, 3, 5 \rangle, \langle 4, 1, -3 \rangle\}$ , find a subset of  $S$  that is a basis for  $\text{span}(S)$ .
  - (b) Extend the set  $S = \{\langle 1, 2, 1, 1 \rangle, \langle -1, 2, 2, 2 \rangle, \langle -2, 1, 2, 2 \rangle\}$  to a basis of  $\mathbb{Q}^4$ .
-

**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose that  $S$  is a maximal linearly-independent subset of a vector space  $V$  (this means that if  $T$  is any linearly-independent subset of  $V$  containing  $S$ , then in fact  $T = S$ ). Prove that  $S$  is a basis of  $V$ .

• Remark: This fact is key to the proof using Zorn's lemma<sup>1</sup> that every vector space has a basis.

---

5. Let  $W$  be a vector space. Recall that if  $A$  and  $B$  are two subspaces of  $W$  then their sum is the set  $A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$ .

- (a) Suppose that  $A \cap B = \{\mathbf{0}\}$ . If  $\alpha$  is a basis for  $A$  and  $\beta$  is a basis for  $B$ , prove that  $\alpha \cup \beta$  is a basis for  $A + B$ .
- (b) Now suppose that  $\alpha$  is a basis for  $A$  and  $\beta$  is a basis for  $B$ . If  $\alpha \cup \beta$  is a basis for  $A + B$ , prove that  $A \cap B = \{\mathbf{0}\}$ .

The situation in (a)-(b) is very important and arises often. Explicitly, if  $A$  and  $B$  are two subspaces of  $W$  such that  $A + B = W$  and  $A \cap B = \{\mathbf{0}\}$  is the trivial subspace, we write  $W = A \oplus B$  and call  $W$  the (internal) direct sum of  $A$  and  $B$ .

- (c) Show that  $\mathbb{R}^2$  is the direct sum of the subspaces given by the  $x$ -axis and the  $y$ -axis.
- (d) Prove that  $W = A \oplus B$  if and only if every vector  $\mathbf{w} \in W$  can be written uniquely in the form  $\mathbf{w} = \mathbf{a} + \mathbf{b}$  where  $\mathbf{a} \in A$  and  $\mathbf{b} \in B$ .
- (e) If  $W = A \oplus B$ , show that  $\dim(W) = \dim(A) + \dim(B)$ . Show using an explicit counterexample that the converse statement need not hold.
- (f) If  $\text{char}(F) \neq 2$ , show that the space  $M_{n \times n}(F)$  is the direct sum of the subspace  $\mathcal{S}$  of symmetric matrices with the subspace  $\mathcal{T}$  of skew-symmetric matrices. [Hint: See problem 7(b) from homework 1.]
- 

6. Let  $V$  be a vector space over  $\mathbb{C}$  such that  $\dim_{\mathbb{C}} V = n$ . Prove that if  $V$  is now considered a vector space over  $\mathbb{R}$  (using the same addition and scalar multiplication), then  $\dim_{\mathbb{R}} V = 2n$ .
- 

7. Let  $F$  be a finite field of cardinality  $q$ . The goal of this problem is to compute the number of invertible  $n \times n$  matrices in  $M_{n \times n}(F)$ .

- (a) Show that the number of invertible  $n \times n$  matrices in  $M_{n \times n}(F)$  is equal to the number of ordered lists  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  of  $n$  linearly independent vectors from  $F^n$ .
- (b) Suppose  $W$  is a  $k$ -dimensional subspace of  $F^n$ . Show that  $W$  contains exactly  $q^k$  vectors.
- (c) For any integer  $0 \leq k \leq n$ , show that there are exactly  $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$  ordered lists  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  of  $k$  linearly independent vectors from  $F^n$ . [Hint: Count the number of ways to choose the vector  $\mathbf{v}_{k+1}$  not in  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ .]
- (d) Deduce that the number of invertible  $n \times n$  matrices in  $M_{n \times n}(F)$  is equal to  $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}) \cdots (q^n - q^{n-1})$ . In particular, find the number of invertible  $5 \times 5$  matrices over the field  $\mathbb{F}_2$ .
- 

<sup>1</sup>Here is the argument: Let  $\mathcal{F}$  be the collection of all linearly-independent subsets of  $V$ , partially ordered by inclusion, and note that  $\mathcal{F}$  is not empty since it contains the empty set. If  $\mathcal{C}$  is any chain in  $\mathcal{F}$  (a subset of  $\mathcal{F}$  such that  $A \subseteq B$  or  $B \subseteq A$  for any  $A, B$  in  $\mathcal{C}$ ), then we claim that  $\mathcal{C}$  has an upper bound given by the union of all the elements of  $\mathcal{C}$ . To see this we simply observe that any linear dependence in the union would imply a linear dependence in one of the elements in the chain (since linear dependences involve only finitely many vectors, we may take the maximum of the subsets in which all vectors appear), contradicting the definition of  $\mathcal{F}$ . Hence the union is linearly independent, so it is also an element of  $\mathcal{F}$ . Then by Zorn's lemma (if  $\mathcal{F}$  is a nonempty partially-ordered set in which every chain has an upper bound, then  $\mathcal{F}$  contains a maximal element),  $\mathcal{F}$  contains a maximal element. Such a maximal element is a maximal linearly-independent subset of  $V$ : then by the result above, it is a basis of  $V$ .